INTERVAL FUNCTIONS AND ABSOLUTE CONTINUITY

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1. Introduction

Suppose $[a, b]$ is a number interval.

The author [1] has shown the following theorem:

THEOREM A. *If each of h and m is a real-valued nondecreasing function on $[a, b]$, and H is a real-valued bounded function of subintervals of $[a, b]$ such that the integral (Section 2)*

$$\int_{[a,b]} H(I)\, dm$$

exists, then the integral

$$\int_{[a,b]} H(I) \int_I (dh)^p\, (dm)^{1-p}$$

exists for each number p such that $0 < p < 1$.

We note that in the above theorem the function w on $[a, b]$ such that

$$w(a) = 0 \quad \text{and} \quad w(x) = \int_{[a,x]} (dh)^p (dm)^{1-p} \quad \text{for} \quad a < x \leqq b,$$

is absolutely continuous with respect to m. This suggests an extension of Theorem A, and in this paper we prove (Theorem 3) that if each of h and m is a real-valued nondecreasing function on $[a, b]$, then the following four statements are equivalent:

(1) If H is a real-valued bounded function of subintervals of $[a, b]$ such that $\int_{[a,b]} H(I)\, dm$ exists, then $\int_{[a,b]} H(I)\, dh$ exists.

(2) $\int_{[a,b]} (dh)^p (dm)^{1-p} \to h|_a^b$ as $p \to 1$ for $0 < p < 1$.

(3) $\int_{[a,b]} \left| dh - \int_I (dh)^p (dm)^{1-p} \right| \to 0$ as $p \to 1$ for $0 < p < 1$.

(4) h is absolutely continuous with respect to m.

2. Preliminary lemmas and definitions

Suppose $[a, b]$ is a number interval.

Throughout this paper all integrals discussed are Hellinger [2] type limits of the appropriate sums, i.e., if K is a real-valued function of subintervals of $[a, b]$, and $[r, s]$ is a subinterval of $[a, b]$, then $\int_{[r,s]} K(I)$ denotes the limit, for successive refinements of subdivisions, of sums $\sum_E K(I)$, where E is a subdivision of $[r, s]$ and the sum is taken over all intervals I of E. We see that $\int_{[a,b]} K(I)$ exists if and only if for each subinterval $[u, v]$ of $[a, b]$, $\int_{[u,v]} K(I)$ exists, so that if $a \leqq u < v < w \leqq b$, then

$$\int_{[u,w]} K(I) = \int_{[u,v]} K(I) + \int_{[v,w]} K(I).$$

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