SOME REPRESENTATION THEOREMS FOR INVARIANT PROBABILITY MEASURES

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Throughout this paper X will be a set, $\mathcal{R}$ will be a σ-algebra of subsets of X (for a definition of σ-algebra and σ-ring of subsets of X see [3]), T will be a mapping of X into X, and m will be a measure on $\mathcal{R}$. We say that $\mathcal{R}$ is *T-invariant* if $A \in \mathcal{R}$ implies $T^{-1}A \in \mathcal{R}$, and a set $A \in \mathcal{R}$ is *T-invariant* if $A = T^{-1}A$. If $\mathcal{R}$ is T-invariant, and if $m(A) = m(T^{-1}A)$ for all $A \in \mathcal{R}$, we say that m is *T-invariant.* We say that m is a probability measure on $\mathcal{R}$ if $m(X) = 1$. If m is a T-invariant probability measure, and if $m(A) = 0$ or 1 for every T-invariant set $A \in \mathcal{R}$, we say that m is *ergodic.* If m is a measure on $\mathcal{R}$ and if $E \in \mathcal{R}$, the measure m_1 defined by $m_1(A) = m(A \cap E)$, all $A \in \mathcal{R}$, is called the *contraction* of m to the set E.

In [1] Blum and Hanson studied the problem of expressing a T-invariant probability measure as a "combination" of some sort of ergodic measures. The following proposition can be inferred from their work.

PROPOSITION 1. *Let T be a 1-1 mapping of X onto X, let $\mathcal{R}$ be a T-invariant σ-algebra of subsets of X, let m be a T-invariant probability measure on $\mathcal{R}$, and let $\mathcal{E}$ be the set of all ergodic measures on $\mathcal{R}$. Suppose that for any T-invariant set $A \in \mathcal{R}$ for which there is a T-invariant probability measure m_0 with $m_0(A) > 0$ there is a $p \in \mathcal{E}$ for which $p(A) > 0$. Then m has an integral representation on $\mathcal{E}$; i.e., there is a probability measure μ on a σ-algebra of subsets of $\mathcal{E}$ such that for any set $A \in \mathcal{R}$, we have that $p(A)$, regarded as a function of p, is measurable on $\mathcal{E}$ and $m(A) = \int_{p \in \mathcal{E}} p(A)\, d\mu$.*

Employing methods similar to those in [1], Farrell [2] studied situations in which X is a topological space and $\mathcal{R}$ consists of the Baire subsets of X. The following proposition can be inferred from the work of Farrell.

PROPOSITION 2. *Let X be a compact Hausdorff space, let $\mathcal{R}$ consist of the Baire subsets of X, and let T be a continuous mapping of X into X. Then any T-invariant probability measure m on $\mathcal{R}$ has an integral representation as in Proposition 1.*

The purpose of the present paper is to construct analogues of Proposition 2 in which X is not required to be compact (or locally compact or σ-compact or metrizable) and to apply these analogues to several concrete examples to which the results stated in [2] are not applicable.

Now let $\mathcal{F}$ be a real vector lattice of bounded real-valued functions on X. We say that $\mathcal{F}$ is *T-invariant* if $f(x) \in \mathcal{F}$ implies $f(Tx) \in \mathcal{F}$. If $\mathcal{F}$ is T-invariant,

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