# AN APPLICATION OF PROHOROV'S THEOREM TO PROBABILISTIC NUMBER THEORY

BY

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The purpose of this paper is to give a probabilistic proof of a theorem which contains as special cases certain results about the natural densities of arithmetically interesting sets of integers, results usually conjectured on probabilistic grounds but proved by nonprobabilistic methods. The principal tool is Prohorov's theorem on the weak compactness of probability measures [8]. Results similar to those of this paper, but for logarithmic density rather than natural density, have been proved, by very different methods, by Paul [7] (see the end of the paper for a comparison).

## 1. Introduction

Let $\mu_N$ be the probability measure on the space of positive integers that places mass $1/N$ at each of the points 1, 2, $\cdots$, $N$. To ask if a set $A$ of integers has a natural density is to ask if the limit $\lim_N \mu_N(A)$ exists. Thus we are lead to ask whether the measures $\mu_N$ converge in some sense. In order to obtain a satisfactory answer, we must first complete the space of integers in some way. The following completion is useful for problems of multiplicative number theory.

Let $X$ be the space of sequences $x = (x_1, x_2, \cdots)$ of nonnegative integers. For each $n \geq 1$ let $\alpha(n) = (\alpha_1(n), \alpha_2(n), \cdots)$, where $\alpha_i(n)$ is the exponent of the $i^{\text{th}}$ prime $p_i$ in the factorization of $n = \prod_i p_i^{\alpha_i(n)}$. The mapping $\alpha$ provides a one-to-one correspondence between the set of positive integers and the subset $X_0$ of $X$ consisting of those $x$ that have only finitely many nonzero coordinates. The completion that we will use is a space $X_\lambda$, between $X_0$ and $X$ ($X_0 \subset X_\lambda \subset X$), defined as follows. For each component $x_i$ of $x$, let $x_i'$ be $x_i$ or 0 according as $x_i \leq 1$ or $x_i > 1$, and let $x_i''$ be $x_i$ or 0 according as $x_i > 1$ or $x_i \leq 1$. For a fixed sequence $\lambda = (\lambda_1, \lambda_2, \cdots)$ of positive constants, let $X_\lambda$ consist of those points $x$ of $X$ for which each of the sums $\sum_i \lambda_i x_i'$ and $\sum_i x_i''$ is finite. (Of course the second sum is finite if and only if at most finitely many of the $x_i$ exceed 1.) Under the metric

$$d(x, y) = \sum_i \lambda_i \,|\, x_i' - y_i' \,|\, + \sum_i |\, x_i'' - y_i'' \,|,$$

$X_\lambda$ is a complete, separable metrix space. (To see this, identify $x$ with $(x', x'')$, where $x'$ and $x''$ have coordinates $x_i'$ and $x_i''$, respectively. With this identification, $X_\lambda$ is the topological product of two sets, the first [second]

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