ON THE ZEROS OF RIESZ' FUNCTION IN THE ANALYTIC THEORY OF NUMBERS

BY
HERBERT S. WILF[1]

In a classical paper [1] M. Riesz introduced the entire function

$$F(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}z^n}{(n-1)!\zeta(2n)} \tag{1}$$

and showed that a necessary and sufficient condition for the truth of Riemann's hypothesis is that for each $\varepsilon > 0$

$$F(x) = O(x^{1/4+\varepsilon}) \qquad (x \to +\infty). \tag{2}$$

Riesz also showed that $F(z)$ is of order one, type one, genus one, has infinitely many zeros off the real axis, at least one on the real axis, has none in the left half-plane and satisfies

$$\textstyle\sum_{n=1}^{\infty} F(z/n^2) = ze^{-z}, \tag{3}$$

$$F(z) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} ze^{-z/n^2} \tag{4}$$

for all z.

In this note we prove certain additional properties of the set of zeros of $F(z)$. Let $\{r_n e^{i\theta_n}\}_1^{\infty}$ denote some arrangement of these zeros in nondecreasing order of modulus, let $x_1, x_2, \cdots$ denote the subsequence of positive real zeros of $F(z)$, and let $h(r, \delta)$ denote the number of zeros in the sector

$$|z| \leqq r, \qquad |\arg z| \leqq \tfrac{1}{2}\pi - \delta \qquad (\delta > 0).$$

Then we show that

$$r_n \sim n\pi \qquad (n \to \infty), \tag{5}$$

$$h(r, \delta) = o(r) \qquad (r \to \infty), \tag{6}$$

$$\textstyle\sum_{n=1}^{\infty} x_n^{-1} < \infty. \tag{7}$$

(8) There are infinitely many x_n and in fact

$$\textstyle\sum_{x_n < x} 1 = \Omega(\log x) \qquad (x \to \infty).$$

The relations (5)–(7) depend hardly at all on the nature of the coefficients $\mu(n)$ in (4) whereas (8) depends on very specific properties of these coefficients.

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