A NONLINEAR INTEGRAL OPERATION[1]

BY

J. S. Mac Nerney

In an earlier paper [4], there was developed a fundamental correspondence between certain additive and multiplicative integration processes, where the integration is directed along intervals in some linearly ordered system, and where the functions involved have their values in a complete normed ring. That development led to an analysis of *linear* integral equations of the form

$$(1) \qquad u(x) = P + (R) \int_x^c V \cdot u,$$

where the right integral is the limit, through successive refinement of subdivisions, of sums of the form $\sum_1^n V(t_{i-1}, t_i)u(t_i)$ [4, p. 155]. This analysis, of the linear case, required only locally bounded variation of the functions involved, thus obviating additional continuity hypotheses of earlier treatments [9], [2], [3], [1] of similar systems. These interrelated treatments are summarized in [4, Sec. 10] (also in [5]).

Now we present an extension to a *nonlinear* situation of the aforementioned fundamental correspondence. This extension leads us to a characterization of solutions u of equations of the form (1), where the linearity hypothesis on the values $V(x, y)$ (of the function V) is replaced by a Lipschitz-type condition. Such a condition, in connection with integrals of the type contemplated here, seems first to have been investigated by J. W. Neuberger [7, p. 542 ff.]. Our results overlap those of Neuberger only in case the underlying system is a linear continuum and all of the functions involved are subjected to additional hypotheses of continuity (see [7, Theorems F and G]).

Throughout this paper, it is to be supposed that S denotes some nondegenerate set, with linear $(\leq)$ ordering $\mathcal{O}$; $\{G, +, \| \ \|\}$ denotes a complete normed Abelian group, with zero element 0; and H denotes the class of all functions from G to G to which $\{0, 0\}$ belongs, with identity function 1. As in [4], we let $\mathcal{O}\mathcal{A}^+$ denote the class of all $\mathcal{O}$-additive functions from $S \times S$ to the set of nonnegative real numbers, and $\mathcal{O}\mathcal{M}^+$ denote the class of all $\mathcal{O}$-multiplicative functions from $S \times S$ to the set of real numbers not less than 1. It should be recalled [4, Theorems 2.2 and 4.3] that there is a reversible function $\mathcal{E}^+$ to which the ordered pair $\{\alpha, \mu\}$ belongs only in case one of the following holds:

(a) α is in $\mathcal{O}\mathcal{A}^+$ and $\mu(x, y) = {}_x\prod^y[1 + \alpha]$ for all $\{x, y\}$ in $S \times S$.
(b) μ is in $\mathcal{O}\mathcal{M}^+$ and $\alpha(x, y) = {}_x\sum^y[\mu - 1]$ for all $\{x, y\}$ in $S \times S$.
(c) $\{\alpha, \mu\}$ is in $\mathcal{O}\mathcal{A}^+ \times \mathcal{O}\mathcal{M}^+$ and, for each $\{x, y\}$ in $S \times S$,

Received July 5, 1963.

[1] Presented to the American Mathematical Society on January 23, 1964.