GENERALISATIONS OF A CLASSICAL THEOREM ABOUT NILPOTENT GROUPS

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1. Introduction

A classical problem in the theory of groups concerned the finite nonabelian groups of which every proper subgroup is abelian. Miller and Moreno considered this problem in 1903—see [10]—and Rédei gave a list of all such groups in his paper [11]. The corresponding problem with "abelian" replaced by "nilpotent" is the subject of [12] and [9], and in the many other generalisations attention has been concentrated on nonnilpotent groups; for instance, Suzuki in [13] showed that any finite simple group with every second maximal subgroup nilpotent has order 60.

The aim of the present paper is to generalise the classical problem in another direction—we study *nilpotent* groups of which every proper subgroup, or every m^{th} maximal subgroup, is nilpotent of given class, and while results can be found for infinite groups we shall keep matters simple by examining the finite case only.

The following facts can be assembled from [10], or deduced from Rédei's list in [11], or proved independently without hardship:

THEOREM. *Let G be a nonabelian group of order p^{r+1} in which every subgroup of order p^r is abelian. Then*

(i) *G is generated by two elements*;

(ii) *G has class 2*;

(iii) *$\gamma_2(G)$ has order p*;

(iv) *$G/\zeta_1(G)$ has order p^2.*

That is the theorem to be generalised. Two other points of interest emerge from Rédei's paper: there are two infinite sequences of these p-groups together with an exceptional 2-group namely the quaternion group of order 8; and ζ_1/γ_2 is unbounded.

In any group the product of normal subgroups with classes n_1 and n_2 respectively has class $n_1 + n_2$; that is a theorem of Fitting [2], and it is a best possible result because in [5], P. Hall gives an example, for any n, which is of class precisely n and which is the product of n abelian normal subgroups. (For what it is worth, we state that there are even metabelian groups with these properties.) Now, since every maximal subgroup of a finite p-group G is normal, G has class $2n$ when every proper subgroup has class n, and we ask when the class is precisely $2n$. This is certainly possible when $n = 1$,

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