ON L-SERIES WITH REAL CHARACTERS

BY

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1. Introduction

Let d be the discriminant of an imaginary quadratic field. Thus there exists a square-free negative integer D with

$$\begin{aligned} d &= D \quad \text{if} \quad D \equiv 1 \pmod 4 \\ &= 4D \quad \text{if} \quad D \equiv 2, 3 \pmod 4). \end{aligned}$$

Such integers d are frequently called fundamental discriminants.

Let

$$\chi_d = \chi_d(n) = \left(\frac{d}{n}\right)$$

be the Kronecker symbol and suppose that

$$L(s, \chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s}$$

is the Dirichlet series associated with the real nonprincipal primitive character $\chi_d \bmod |d|$.

The behaviour of $L(s, \chi_d)$ for real s between 0 and 1 has important implications in the study of the class number $h(d)$ of quadratic fields of discriminant d. In particular the existence or nonexistence of roots of $L(s, \chi_d)$ in the interval $0 < s < 1$ has far-reaching consequences.

A conjecture, in milder form due to Hecke, states that if $0 < s < 1$, then $L(s, \chi_d) \neq 0$. This conjecture is still unsettled.

The object of this note is to examine the mean value of $L(s, \chi_d)$ summed over fundamental discriminants. In particular our object is to prove the following

THEOREM. *If d is a fundamental discriminant and $\chi_d(n)$ the associated Kronecker symbol, then for $\frac{1}{2} < s \leqq 1$, we have*

$$\sum_{0<-d\leqq N} L(s, \chi_d) = N \frac{\zeta(2s)}{\zeta(2)} \prod_p \left(1 - \frac{1}{(p+1)p^{2s}}\right) + O\left(\frac{N^{(2/3)(2-s)} \log N}{2s-1}\right),$$

where the summation is over fundamental discriminants and the constant implied by the O is absolute.

This leads immediately to the following

COROLLARY. *For any given s in the interval $\frac{1}{2} < s \leqq 1$, there exists $N_0 = N(s)$*

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