THE LIE ALGEBRAS WITH A NONDEGENERATE TRACE FORM

BY

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1. Introduction

A trace form on a Lie algebra L is a bilinear form f on L for which there is a representation Δ of L, of finite degree, such that

$$f(a, b) = \operatorname{tr}(\Delta a \Delta b) \qquad (a, b \in L).$$

In this paper we shall show that if L is a Lie algebra over an algebraically closed field F of characteristic $p > 3$, and if L has a nondegenerate trace form, then L is a direct sum of Lie algebras which are either one-dimensional, isomorphic to a total matrix algebra $M_n(F)$ with n a multiple of p, or simple of classical type but not of type PA. The simple algebras of classical type were classified by Mills and Seligman in [3]; they are the analogues (as described, for example, in [4]) over F of the complex simple algebras (including the five exceptional algebras). Included among the simple algebras of classical type are the algebras of *type* PA, a Lie algebra L over F being said to be of type PA if for some multiple n of p, $L \cong PSM_n(F)$, the Lie algebra of all $n \times n$ matrices of trace 0, modulo scalar matrices.

Conversely, it is known that all of the direct summands mentioned above do have a nondegenerate trace form, with the possible exception of the algebra of type E_8 when $p = 5$.

It is to be hoped that the results of this paper will be applicable to the theory of finite groups. For the relationship to that subject, see [6]. Actually, for this application, one is concerned with the case in which the base field F is finite. Theorem 5.1 below gives the structure of L over finite fields; the actual classification of the algebras in this case is, however, a quite different topic.

2. Preliminaries

If f is a trace form on a Lie algebra L, we shall denote by $L^\perp$ the radical of f, that is, the set of all a in L such that $f(a, b) = 0$ for all b in L. Now $L^\perp$ is an ideal of L and f induces a bilinear form $\bar{f}$ on the quotient algebra $\bar{L} = L/L^\perp$. By a *quotient trace form* on a Lie algebra $\bar{L}$ is meant any bilinear form $\bar{f}$ arising in this way from a trace form f on an algebra L such that $\bar{L} = L/L^\perp$. Thus a quotient trace form is in particular a nondegenerate symmetric invariant form.

It has been shown by Block [1] that if L is a simple Lie algebra over an algebraically closed field F of characteristic $p > 3$, and if L has a quotient trace form, then L is of classical type. The algebras of type PA have a

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