CONVEX POLYTOPES IN LINEAR SPACES

BY

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1. Introduction

H. Weyl [7] defines a convex polyhedron as a subset P, of a finite dimensional space E^n, which can be expressed as the intersection of a finite number of half spaces. Here we generalize this definition in a meaningful way to infinite dimensional spaces. In Section 1, it is shown that such sets have "faces" and in general enjoy many of the geometric properties of their finite dimensional counterparts. However, as Theorem 2.4 points out, nondegenerate bounded convex polytopes in infinite dimensional spaces do not have any extremal points. This enables us to prove that reflexive Banach spaces do not contain any bounded convex polytopes.

In Section 3, a comparison is made between out convex polytopes and those defined by Bastiani [1]. Although a direct comparison is not possible, we show that our sets essentially satisfy her definition but produce a counter example to show that the converse is not true.

Clarkson [3] and Fullerton [5] among others have characterized B-spaces by the shapes of their unit spheres. Here we extend this work showing that subspaces of the B-space (c_0) of all sequences which converge to zero are, and are the only, separable B-spaces whose unit spheres are convex polytopes. The space (c_0) itself is the only B-space whose unit sphere is a "paralleletope" (a generalized parallelepiped). This generalizes and reproves a result of Klee [6]. Namely that every symmetric convex polytope can be realized as the central section of the unit ball of (c_0).

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2. Definition and properties of convex polytopes

Throughout this section, X will denote a real locally convex linear topological space which is Hausdorff and P a convex subset of X such that the origin θ is in the interior P^0, of P. If $\{E_\alpha \mid \alpha \in \mathrm{A}\}$ is a collection of closed half spaces such that $P = \bigcap \{E_\alpha : \alpha \in \mathrm{A}\}$ and if for each $x \in X$ there exists a finite subcollection $\alpha_1, \alpha_2, \cdots, \alpha_k$ of A having the property that

$$x \in \bigcap \{E_\alpha \mid \alpha \in \mathrm{A}, \alpha \neq \alpha_1, \alpha_2, \cdots, \alpha_k\},$$

then P will be called a *convex polytope.* For the rest of this section, we will

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