ON THE UNITS OF AN ALGEBRAIC NUMBER FIELD

BY

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Introduction

Let K be an algebraic number field of degree n over the field of rational numbers Q. Let p be a rational prime and denote the p-adic completion of Q by Q_p. Let A denote the completion of the algebraic closure of Q_p equipped with its valuation $|\ \ |_p$ normed so that $|p|_p = 1/p$. Let T be the set of n distinct monomorphisms of K into A.

The p-adic rank $r_p = r_{K,p}$ of the units U of K is defined as the rank of the p-adic regulator matrix

$$\mathcal{R}_p = (\log_p \tau(V_i))_{\tau \in T, i=1,\cdots,r}$$

where $v_1, \cdots, v_r$ is a basis for a free direct summand of U of maximal rank ($r = r_K =$ dirichlet number of K) and where the p-adic logarithm is defined by the usual series for principal units and extended to all units of A by means of the functional equation. Thus if $v \in A$ is such that $|v - 1|_p < 1$ then $\log_p v = -\sum_{k=1}^{\infty} (1 - v)^k/k \in A$ and if $|v|_p = 1$ then

$$\log_p v = (\log v^m)/m$$

for any positive integer m such that $|v^m - 1|_p < 1$.

We have $r_p \leq r$. In the abelian case Leopoldt in [6] has raised the question of determining r_p and in particular asked if $r_{K,p} = r_K$ for all abelian K and rational primes p. In §1 we prove the following partial result on Leopoldt's problem.

THEOREM 1. *If K/Q is an abelian extension with galois group G of exponent m such that $m \leq 4$ or $m = 6$, then $r_p = r$.*

The proof uses Mahler's p-adic analogue [7], [8] of Hilbert's seventh problem (α^β is transcendental if α and β are algebraic numbers such that $\alpha \neq 0, 1$ and β is irrational). The same proof actually proves a slightly stronger result (Theorem 1′) as well as the following fact.

THEOREM 2. *If K/Q is normal and $r \geq 2$ then $r_p \geq 2$.*

In §2 an algebraic method is employed to solve the following special cases of Leopoldt's problem.

THEOREM 3. *Let p be a regular prime, let a be a positive integer, let ζ be a primitive p^a-th root of unity and let $K = Q(\zeta)$. We then have $r_p = r$.*

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