ON HOMOTOPY 3-SPHERES[1]

BY

WOLFGANG HAKEN

A homotopy 3-sphere M^3 is a compact, simply connected 3-manifold without boundary. After the work of Moise [6] and Bing [1] M^3 possesses a triangulation. The Poincaré conjecture [9] states that every homotopy 3-sphere M^3 is a 3-sphere. In this paper we prove three theorems, related to the Poincaré conjecture, about maps of a 3-sphere S^3 onto M^3 and about 1- and 2-spheres in M^3.

1. Theorems 1 and 2, concerning maps $S^3 \to M^3$ and closed curves in M^3. From the work of Hurewicz [5], Part III, it follows that there exists a continuous map $\varphi : S^3 \to M^3$ of degree 1 (where S^3 means a 3-sphere). We shall prove that there exists an especially simple map of this kind.[2]

THEOREM 1. *If M^3 is a homotopy 3-sphere then there exists a simplicial map $\gamma : S^3 \to M^3$ of degree 1 such that the singularities of γ (i.e. the closure of the set of those points $p \in M^3$ for which $\gamma^{-1}(p)$ consists of more than one point) lie in a (polyhedral, compact) handlebody in M^3.*

One might consider this result as a step towards a proof of the Poincaré conjecture. Indeed, if it were possible to restrict the singularities of γ to a 3-cell in M^3 instead of a handlebody the existence of a homeomorphism $S^3 \to M^3$ would follow.

From Theorem 1 we may derive another aspect of the Poincaré problem by considering simple closed curves in M^3.

From the definition of simple connectedness it follows that every closed curve $C^1 \subset M^3$ bounds a singular disk $D^2 \subset M^3$. If C^1 is a tame, simple closed curve then one can find a D^2 which is also tame and possesses only "normal" singularities (see [7], [8]), i.e. double curves in which two sheets of D^2 pierce each other, triple points in which three sheets pierce each other, and branch points from each of which one or more double arcs originate; the triple points, the branch points, and the interiors of the double curves are disjoint from the boundary $\dot{D}^2$ of D^2, but the double curves may have end points in $\dot{D}^2$.

As Bing [2] has proved, M^3 is a 3-sphere if (and only if) every tame, simple closed curve $C^1 \subset M^3$ lies in a (compact) 3-cell in M^3. The statement that C^1 lies in a 3-cell $D^3 \subset M^3$ is equivalent to the statement that C^1 bounds a "knot projection cone" D^2 in M^3, i.e. a (tame) singular disk whose singularities are one branch point P and double arcs originating from P, being pairwise

Received April 10, 1965.

[1] This research was partially supported by the Air Force Office of Scientific Research.

[2] Theorem 1 is a consequence of a "monotonic mapping theorem" announced by Moise in [6a]; however the proof is different from Moise' proof.