THE REPRESENTATION LATTICE OF A LOCALLY COMPACT GROUP

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We begin by defining the term "representation lattice" used in the title. Let G be a separable locally compact group. By a representation of G we shall always mean a strongly continuous homomorphism of G into a group of unitary operators acting on a (not necessarily separable) Hilbert space. Two representations L and M of G are said to be *disjoint*, denoted $L \mathbin{\delta} M$, *if* no subrepresentation of L is (unitary) equivalent to any subrepresentation of M. We say L *covers* M, denoted $L \rbrace M$, if no subrepresentation of M is disjoint from L. We say L is *quasi-equivalent* to M, denoted $L \sim M$, if L covers M and M covers L. (For all of these concepts, see [10] and [11].) The collection Q of all quasi-equivalence classes of representations of G forms a complete distributive lattice with respect to the ordering given by the covering relation. The lattice Q ($= Q(G)$) will be called the *representation lattice of* G The collection Q_σ of all quasi-equivalence classes of separable representations of G forms a σ-complete sublattice of $Q(G)$, which we call the *separable representation lattice of* G.

The properties of quasi-equivalence and the covering relation (cf. [10], [11] and Proposition 1 of [6]) are reminiscent of a projection lattice. Our first theorem proves that this is not a mere impression. $Q(G)$ is lattice isomorphic to the lattice of all projections in an abelian von Neumann algebra. The appropriate von Neumann algebra is just the center of the big group algebra $\mathfrak{A}(G)$, introduced in [7].

There has been a rather extensive study of Q_σ, using the tool of direct integral decomposition theory for representations. (Recall that a group G is said to be type I if it admits only type I representations. A representation is type I if its range generates a type I von Neumann algebra.) In the case where G is type I, G. W. Mackey has characterized $Q_\sigma(G)$ as being lattice isomorphic to the lattice of all standard σ-finite measure classes on the dual $\hat{G}$ of G. (Cf, [12].) ($\hat{G}$ denotes the set of unitary equivalence classes of separable irreducible representations of G.) The measures on $\hat{G}$ arise from the central decomposition of separable multiplicity free representations, as a direct integral of irreducible representations. If G is not type I, the lattice $Q_{\sigma-I}$ of all quasi-equivalence classes of separable type I representations of G is lattice isomorphic to some lattice $\mathfrak{L}_I$ of standard σ-finite measure classes on the dual $\hat{G}$. However $\mathfrak{L}_I$ is a proper sublattice of the lattice of all standard σ-finite measure classes on $\hat{G}$. The characterization of $\mathfrak{L}_I$, when G is not type I, has remained an open problem.

In [6], the author presented a generalization of the Mackey decomposition

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