ON THE ORDER TOPOLOGY IN A LATTICE

BY
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Introduction

Order convergence in a lattice S may fail to be pretopological; when pretopological, it may fail to be topological; when topological, S may fail to be a topological lattice under its order topology.

We first state a condition c_1 on the local ideal structure of S which is necessary and sufficient to make order convergence pretopological. Under c_1, the neighborhood filter at each point x in S is generated by a certain family of closed intervals. Next we state conditions c_2 and c_3 each of which, when combined with c_1, suffices to make order convergence topological. The order topology obtained under c_2 differs in a significant way from that obtained under c_3. In both cases, however, S is a topological lattice under its order topology, and, in each case, the order topology has an open subbase of ideals and dual ideals reminiscent of Frink's *ideal topology* [3] in a lattice.

1. Notation

For a subset $A \subset S$, A^* will denote the set of all upper bounds of A, and A^+ the set of lower bounds of A. $\{x\}^*$ and $\{x\}^+$ will be written x^* and x^+ respectively. The "closed interval" notation, $[x, y] = x^* \cap y^+$ $(x \leq y)$ will be employed.

If A, B are subsets of S, we define

$$A \vee B = \{x \vee y : x \in A,\ y \in B\}; \qquad A \wedge B = \{x \wedge y : x \in A,\ y \in B\}.$$

$\mathfrak{F}$ and $\mathfrak{G}$ will be the usual notation for filters on S. Let

$$\mathfrak{F}^+ = (\bigcup F^+ : F \in \mathfrak{F}) \quad \text{and} \quad \mathfrak{F}^* = (\bigcup F^* : F \in \mathfrak{F}).$$

One can show that the families

$$\{F \vee G : F \in \mathfrak{F}, G \in \mathfrak{G}\} \quad \text{and} \quad \{F \wedge G : F \in \mathfrak{F}, G \in \mathfrak{G}\}$$

are filter bases on S; the filter generated by the first is called $\mathfrak{F} \vee \mathfrak{G}$, and by the second, $\mathfrak{F} \wedge \mathfrak{G}$.

The following relationships are easily verified:

(a) $(\mathfrak{F} \vee \mathfrak{G})^* = \mathfrak{F}^* \vee \mathfrak{G}^*$;
(b) $(\mathfrak{F} \wedge \mathfrak{G})^+ = \mathfrak{F}^+ \wedge \mathfrak{G}^+$;
(c) $(\mathfrak{F} \wedge \mathfrak{G})^* \supset \mathfrak{F}^* \wedge \mathfrak{G}^*$;
(d) $(\mathfrak{F} \vee \mathfrak{G})^+ \supset \mathfrak{F}^+ \vee \mathfrak{G}^+$.

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