# GENERALIZATION OF A FORMULA OF HAYMAN AND ITS APPLICATION TO THE STUDY OF RIEMANN'S ZETA FUNCTION

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**1. Introduction.** In [5] Hayman considers functions $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$, analytic inside $|z| < R$ ($\leq \infty$) and satisfying some additional conditions and obtains for their coefficients $\alpha_n$ an asymptotic estimate which generalizes Stirling's formula $1/n! \sim (e/n)^n(2\pi n)^{-1/2}$, to which it reduces in the case $f(z) = e^z$.

In the first part of the present paper, we obtain an asymptotic series for the coefficients $\alpha_n$ of an appropriate class of functions $f(z)$; this is the analog of the well-known asymptotic series for the factorials, to which it reduces in the case $f(z) = e^z$.

In the second part, we use the results of the first part, in order to prove that a certain condition, necessary for the validity of the Riemann hypothesis, is in fact satisfied.

## Part I

**2. Notations and main results.** Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function analytic inside the circle $|z| < R$ ($\leq \infty$), real on the real axis and such that $\lim_{x \to R} f(x) = \infty$. Set

$$a_1(z) = z \frac{d(\log f(z))}{dz} = z(f'(z)/f(z))$$

and define inductively for $\nu > 1$,

$$a_\nu(z) = z \frac{da_{\nu-1}(z)}{dz}.$$

Let $A$ be the class of real-valued functions $a(x)$ such that, for $\nu \geq 3$, $a_\nu(x) \leq a(x)$ for sufficiently large $x$ (which might depend on $\nu$). We assume furthermore, that there exists a function $\delta(x)$, satisfying the following conditions for some $a \in A$:

(i) $\lim_{x \to R} \delta_2(x) a_2(x) = \infty$;
(ii) $\lim_{x \to R} \delta^3(x) a(x) = 0$;

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