A TRANSPLANTATION THEOREM BETWEEN ULTRASPHERICAL SERIES[1]

BY

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1. Introduction

In the introduction we shall describe our results for Legendre and cosine series. Analogous results hold for ultraspherical series but in the interest of simplicity we state them here only in the most important special case.

$P_n(x)$ is the Legendre polynomial of degree n. The functions

$$(n + \tfrac{1}{2})^{1/2} P_n(\cos\theta)(\sin\theta)^{1/2}$$

are orthonormal functions on $(0, \pi)$. They also have the known asymptotic formula [16, Th. 8.21.5]

$$(n + \tfrac{1}{2})^{1/2} P_n(\cos\theta)(\sin\theta)^{1/2} = A \cos[(n + \tfrac{1}{2})\theta - \pi/4] + O(1/(n \sin\theta)),$$

$0 < \theta < \pi$. Classically this has been used to set up equiconvergence theorems between Legendre series and cosine series, but only for

$$0 < \varepsilon \leqq \theta \leqq \pi - \varepsilon < \pi.$$

While it isn't possible to get uniform equiconvergence theorems for $0 \leqq \theta \leqq \pi$, it is possible to get a theorem that uses all θ, $0 \leqq \theta \leqq \pi$.

Let $f(\theta)$ be a function in $L^{p,\alpha}(0, \pi)$ where $L^{p,\alpha}$ is the class of measurable functions for which

$$\|f\|_{p,\alpha} = \left[\int_0^\pi |f(\theta)|^p (\sin\theta)^{\alpha p}\, d\theta\right]^{1/p}$$

is finite. In all that follows we will have

$$1 < p < \infty \quad \text{and} \quad -1/p < \alpha < 1 - 1/p.$$

These are the familiar conditions that are necessary to have the Hilbert transform a bounded operator. Also, if $f \in L^{p,\alpha}$ then $f \in L^{1,0}$, so we may talk about its Fourier series. Let

$$a_n = \frac{1}{\pi} \int_0^\pi f(\theta) \cos n\theta\, d\theta. \tag{1}$$

Then

$$f(\theta) \sim a_0/2 + \textstyle\sum_{n=1}^\infty a_n \cos n\theta.$$

Since $(n + \frac{1}{2})^{1/2} P_n(\cos\theta)(\sin\theta)^{1/2}$ behaves about like $\cos n\theta$ we set

$$T_r f(\theta) = \textstyle\sum_{n=0}^\infty a_n r^n (n + \frac{1}{2})^{1/2} P_n(\cos\theta)(\sin\theta)^{1/2}$$

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