GENERALIZED GROUP ALGEBRAS

BY
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1. Introduction

E. Hewitt and H. Zuckerman have shown how to define convolution multiplication in a very general context [7]. In particular, using their multiplication it is possible to make the conjugate space of the complex Banach space of all bounded complex-valued functions defined on a semigroup into a Banach algebra. This algebra has been studied previously by M. M. Day [2] in case the semigroup is left amenable. This algebra, however, seems ill-suited to the study of harmonic analysis due both to its size and to the lack of available analytical machinery.

We propose to continue the study of "harmonic analysis" in the context of left amenable groups but with two inovations. Firstly, we utilize the Stone-Čech compactification of the discrete semigroup to place our study in the context of regular Borel measures on a compact Hausdorff space, cf. [3]. Secondly, we restrict our attention to the L_2 space of the measure "associated" with a left invariant mean. We show that this is also a Banach algebra under convolution multiplication and this is the generalized group algebra referred to in the title. One of our interests in this group algebra results from its connection with several questions we raised in [3]. The relation of our work with these questions is discussed in §5.

Our utilization of the Stone-Čech compactification is given in §2 along with other preliminaries. In §3 the convolution multiplication is defined and some properties of it are derived. The generalized group algebra is defined in §4 and some of its structure (including the determination of its Jacobson radical) is derived in §5. We conclude with some remarks in §6.

2. Preliminaries

Let Σ be a semigroup. We shall denote by $\mathfrak{B}(\Sigma)$ the complex Banach space of bounded complex-valued functions on Σ in which

$$\|f\| = \sup\{|f(\sigma)| \mid \sigma \in \Sigma\}.$$

An element L of $\mathfrak{B}(\Sigma)^*$ (the conjugate space of $\mathfrak{B}(\Sigma)$) is said to be a left invariant mean on Σ if (1) $\|L\| = 1$; (2) $Lf \geq 0$ for $f \geq 0$; and (3) $L({}_\sigma f) = Lf$ for each $\sigma \in \Sigma$, where $({}_\sigma f)(\tau) = f(\sigma\tau)$ for $\tau \in \Sigma$. Not every semigroup possesses a left invariant mean; a semigroup that does is said to be left amenable. An abelian semigroup is always left amenable. For this result and further information on left invariant means see [6, §17].

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