PRINCIPAL SUBMATRICES OF NORMAL AND HERMITIAN MATRICES

BY

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1. Introduction

In this paper we obtain inequalities and location theorems linking all the eigenvalues of all of the principal $k \times k$ submatrices of a normal or Hermitian $n \times n$ matrix A to the eigenvalues of A. We also obtain inequalities for certain expressions involving $k \times k$ subdeterminants of A. In addition we examine the possible occurrences of a multiple eigenvalue of A among the eigenvalues of the principal $k \times k$ submatrices of A. Certain of our theorems for normal matrices hold only when $k = n - 1$. It is an interesting and open question to find analogues of these theorems for $k \times k$ principal submatrices. For Hermitian matrices we obtain stronger theorems than are possible for arbitrary normal matrices. In one of our theorems (Theorem 3) we only require that A be diagonable.

2. Notation

In this paper $A = (A_{ij})$ denotes an $n \times n$ diagonable matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Usually A will be normal. In general the eigenvalues are not all distinct so let $\mu_1, \mu_2, \cdots, \mu_s$ denote the distinct eigenvalues, where the multiplicity of μ_i is e_i for $1 \leqq i \leqq s$; $e_1 + \cdots + e_s = n$.

We arrange the notation so that

$$(\lambda_1, \lambda_2, \cdots, \lambda_n) = (\mu_1, \cdots, \mu_1, \mu_2, \cdots, \mu_2, \cdots, \mu_s, \cdots, \mu_s).$$

When A is Hermitian we assume $\mu_1 < \mu_2 < \cdots < \mu_s$.

For fixed integers n and k, $1 \leqq k < n$, Q_{nk} denotes the set of all sequences $\omega = \{i_1, i_2, \cdots, i_k\}$ of integers such that $1 \leqq i_1 < i_2 < \cdots < i_k \leqq n$. We always let

$$\omega = \{i_1, i_2, \cdots, i_k\} \quad \text{and} \quad \tau = \{j_1, j_2, \cdots, j_k\}$$

be two typical elements of Q_{nk}. The $k \times k$ matrix B defined by

$$B_{\alpha\beta} = A_{i_\alpha j_\beta}, \qquad 1 \leqq \alpha, \beta \leqq k,$$

is denoted by $A[\omega \mid \tau]$. The $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A is denoted by $A(i \mid j)$. We let $f(\lambda)$, $f_{[\omega]}(\lambda)$, $f_{(i)}(\lambda)$ stand for the characteristic polynomials of A, $A[\omega \mid \omega]$, $A(i \mid i)$, respectively. We let

$$f_{[\omega]}(\lambda) = \lambda^k - c_{\omega 1}\lambda^{k-1} + c_{\omega 2}\lambda^{k-2} - \cdots + (-1)^k c_{\omega k}.$$

Here, of course, $c_{\omega j}$ is the sum of the principal $(k-j) \times (k-j)$ subdetermi-

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