PERMUTABLE CONGRUENCES IN A LATTICE[1]

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The purpose of this paper is to give certain conditions which are equivalent to the permutability of congruences in lattices and to reveal their utility by proving certain results as direct consequences of them. Thus we formulate the conditions which are equivalent to the permutability of two congruence relations on a lattice in Theorem 1. Making use of these conditions we prove in Theorem 2 that any two p-neutral congruences on a lattice permute. (A congruence of the form $x \theta y$ if and only if $x + a = y + a$ for some $a \theta 0$ is said to be a p-neutral congruence on the lattice (cf. [5]).) As corollaries to Theorem 2 we get the known results—any two standard congruences on a lattice permute, any two congruences on a weakly complemented lattice permute and any two congruences on a relatively complemented lattice permute. Further these conditions enable us to give a proof of the result—any two congruences on a distributive lattice permute if and only if L is relatively complemented.

In Theorem 3 we prove: any two congruences on a discrete modular lattice are permutable if and only if L satisfies condition (α); where (α) says that for all a, b, c in L with $a \succ b \succ c$ either (a, b) is projective with (b, c) or there exists a complement d of b in (a, c).($a \succ b$ means a covers b).

Theorem 4 is a generalization of Theorem 3 to weakly modular lattices and proves that any two congruences on a semi-discrete, weakly modular lattice L are permutable if and only if L satisfies condition (β); for all $a > b > c$ either (b, c) is a lattice translate (cf. [8]) of (a, b) or there exists a complement d of b in (a, c).

It is well known that two congruence relations θ and ϕ on a lattice L are said to be permutable if and only if $a \theta b$; $b \phi c$ implies the existence of a d in L such that $a \phi d$; $d \theta c$.

Next we give the conditions which are necessary and sufficient for the permutability of two congruence relations on a lattice.

THEOREM 1. *The following conditions on a lattice L are equivalent.*

(i) *The two congruences θ and ϕ on L permute.*

(ii) *For every comparable pair of elements (a, c), $a \theta b$; $b \phi c$ imply the existence of a d in L with $a \phi d$; $d \theta c$.*

(iii) *For all triples (a, b, c) forming a chain in that order $a \theta b$; $b \phi c$ imply the existence of a d in L with $a \phi d$; $d \theta c$.*

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