ON THE VANISHING OF TOR IN REGULAR LOCAL RINGS

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Introduction

The object of this paper is to provide a proof of the following conjecture of Auslander [1, p. 636]: Let C be a regular local ring, M and N C-modules of finite type. Then $\text{Tor}_i^C(M, N) = 0$ implies $\text{Tor}_j^C(M, N) = 0$ for $j \geq i$. This was previously known only when C is an unramified or equicharacteristic local ring. The proof uses two theorems of some independent interest, concerning the non-negativity of higher Euler characteristics.

The following notations and conventions are used throughout: If A is a ring, and M an A-module, $l(M)$ denotes the length of M. If M and N are two A-modules, we define

$$\chi_j^A(M, N) = \textstyle\sum_{i \geq j} (-1)^{i-j} l(\text{Tor}_i^A(M, N)).$$

The use of these notations presupposes, in the first case, that M is an A-module of finite length, and in the second case, that $\text{Tor}_i^A(M, N)$ is a module of finite length for $i \geq j$ and is 0 for i sufficiently large.

If A is a noetherian ring, and M an A-module of finite type, Supp M (the support of M) is the (closed) subset of Spec A consisting of all prime ideals p of A such that $M_p \neq 0$. The dimension of M (dim M) is the dimension of the noetherian topological space Supp M. We make the convention that dim $(0) = -1$. $\hat{\text{Tor}}(M, N)$ denotes the "complete Tor." See [3 Chapter V] for details.

Statement and Proof of results

LEMMA 1. *Let A be a noetherian local ring with maximal ideal m. Let $x_1, x_2, \cdots, x_d$ be an A-sequence contained in m generating an ideal I. Let M be an A-module of finite type. Assume that M/IM is an A-module of finite length. Then $\text{Tor}_i^A(A/I, M)$ is an A-module of finite length for $i \geq 1$ which is zero for i large, and $\chi_0^A(A/I, M) \geq 0$, with the equality holding iff* dim $M < d$.

Proof. Since Supp $(\text{Tor}_i^A(A/I, M))$ is included in Supp $(M/IM) = \{m\}$, it is clear that the $\text{Tor}_i^A(A/I, M)$ have finite length. The resolution of A/I by the Koszul complex with respect to $x_1, \cdots, x_d$ shows that A/I has finite homological dimension. The rest of the proof proceeds by induction on d. If $d = 0$, the statement is obvious. So assume $d \geq 1$, and let $B = A/x_1$, let $x = x_1$. Then we have the spectral sequence

$$\text{Tor}_p^B(A/I, \text{Tor}_q^A(B, M)) \Rightarrow \text{Tor}_{p+q}^A(A/I, M).$$

Since B has homological dimension 1 as an A-module, the spectral sequence degenerates into an exact sequence:

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