A NOTE ON CERTAIN ARITHMETICAL CONSTANTS

BY

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In their famous memoir "Partitio Numerorum III" [5] Hardy and Littlewood formulated several conjectures about the asymptotic distribution of primes of various special forms. One of them was:

Conjecture K. *If c is any fixed integer other than a cube, then there are infinitely many primes of the form* $m^3 + c$. *The number* $P(N)$ *of such primes up to N is given asymptotically by*

$$P(N) \sim \frac{N^{1/3}}{\log N} \prod_p \left(1 - \frac{2}{p-1} (-c)_p\right), \tag{1}$$

where p runs through primes $\equiv 1 \pmod 3$ *with* $p \nmid c$, *and* $(-c)_p$ *is* 1 *or* $-\frac{1}{2}$ *according as* $-c$ *is or is not a cubic residue* $(\operatorname{mod} p)$.

The problem of computing, for a particular c, the constant given by the product on the right of (1), and similar problems for more general conjectures, have engaged the attention of several mathematicians [1], [2], [3], [12]. A more general conjecture made by Bateman and Horn ([1]; see also [3]) was the following:

Hypothesis H. *Let* $f_1(x), \cdots, f_k(x)$ *be distinct polynomials in one variable with integral coefficients and with highest coefficients positive, of degrees* $h_1, \cdots, h_k$ *respectively. Suppose that each of these polynomials is irreducible over the rational field and that there is no prime which divides* $f_1(n) \cdots f_k(n)$ *for all n. Let* $Q(N)$ *denote the number of positive integers n up to N for which* $f_1(n), \cdots, f_k(n)$ *are all primes. Then*

$$Q(N) \sim (h_1 \cdots h_k)^{-1} C(f_1, \cdots, f_k) \int_2^N (\log u)^{-k}\, du, \tag{2}$$

where

$$C(f_1, \cdots, f_k) = \prod\nolimits_p \{(1 - p^{-1})^{-k}(1 - p^{-1}\omega(p))\}. \tag{3}$$

Here the product is over all primes and $\omega(p)$ *denotes the number of solutions of the congruence*

$$f_1(x) \cdots f_k(x) \equiv 0 \pmod p.$$

This hypothesis implies Conjectures B, D, E, F, K, P of Hardy and Littlewood (cf. [11]).

Bateman and Horn showed that the convergence of the product (3) follows easily from the Prime Ideal Theorem. A similar deduction had been made

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