INVARIANT SUBSPACES

BY

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1. Introduction

The invariant subspaces of L^p $(1 \leq p < \infty)$ of the unit circle are those closed subspaces $\mathfrak{M}$ (weak*-closed when $p = \infty$) for which $e^{ix}f$ is in $\mathfrak{M}$ for each f in $\mathfrak{M}$. If $\mathfrak{M}$ is invariant and also $e^{-ix}f$ is in $\mathfrak{M}$ for every f in $\mathfrak{M}$, $\mathfrak{M}$ is called doubly invariant; $\mathfrak{M}$ is called simply invariant if it is invariant but not doubly invariant. The structure of these subspaces is known. In their fundamental paper [4], Helson and Lowdenslager used an elegant Hilbert space argument to characterize the simply invariant subspaces of L^2. Forelli [2] extended their result to L^1 by a factoring process that depended on the L^2 case. In the remaining cases [3, p. 26], the structure of the simply invariant subspace $\mathfrak{M}$ results from an analysis of $\mathfrak{M} \cap L^2$ $(1 < p < 2)$ and of the annihilator of $\mathfrak{M}$ in $(L^p)^*$ $(2 < p \leq \infty)$. Thus the structure of the simply invariant subspaces of L^p unfolds only after initial success in the L^2 setting. Much the same situation holds for doubly invariant subspaces, and for invariant subspaces defined in certain abstract spaces [8], [9], [10].

The primary purpose of this paper is to obtain these invariant subspace structure theorems by methods that are free of special Hilbert space techniques. We are successful in all cases except one—the simply invariant subspaces in L^1. In §2 we characterize the simply invariant subspaces of $L^p(dm)$ $(1 \leq p < \infty)$ of a Dirichlet algebra by a method that depends on the reflexivity of the overlying function space $(1 < p < \infty)$ and on a double extremal technique developed by Rogosinski and Shapiro [7]. The same method is implicit in an abstract of E. Bishop [Notices, vol. 12 (1965), p. 123]. In §3 we use a Zorn's lemma argument to characterize the doubly invariant subspaces in $L^p(d\mu)$ $(1 \leq p \leq \infty)$ of a certain measure space. Finally, in §4, we give new proofs and expand on some results of Srinivasan and Hasumi [10] concerning weak*-density of subalgebras of $L^\infty(d\mu)$. Although most of the paper is devoted to new proofs of known results, we believe that Theorem 1 $(1 < p < 2, 2 < p < \infty)$ is new.

As one would expect, the technique described in §2 does not apply to the simply invariant subspaces on the line; neither does it seems to offer great promise in the study of invariant subspaces in the spaces of functions from the unit circle into a Hilbert space [3, Lectures V, VI].

2. Simply invariant subspaces

Let X be a compact Hausdorff space, A a uniformly closed algebra of complex, continuous functions on X that contains the constant functions, separates the points of X and satisfies the additional condition that Re A is uniformly

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