ON THE DISCREPANCY OF CERTAIN SEQUENCES MOD 1

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1. Introduction

For any set of N points in the unit square, and any point (x, y) in the square, let $S(x, y)$ denote the number of points of the set which are in the rectangle $0 \leq \zeta < x, 0 \leq \eta < y$. We shall call the difference $\Delta(x, y) = S(x, y) - Nxy$ the error at (x, y). We shall call the quantity $D = \sup |\Delta(x, y)|$, where the supremum is taken over all points (x, y) of the unit square, the discrepancy of the set of points.

K. F. Roth [3] proved that for any set of N points in the unit square, $D > c\sqrt{\log N}$, where c is a positive absolute constant. An analogous result holds in k-dimensional space: $D_k > c_k(\log N)^{(k-1)/2}$, where c_k is a positive constant depending only on k, and D_k is the discrepancy of a set of N points in the k-dimensional unit cube.

Roth also showed an example of 2^n points in the unit square for which the discrepancy $D \leq 2n + 1$. In 1960 J. H. Halton [2] studied a generalization of that example in the k-dimensional unit cube, and obtained an analogous result: $D_k \leq c_k(\log N)^{k-1}$.

Halton also considered the gap between $(\log N)^{(k-1)/2}$ in Roth's theorem, and $(\log N)^{k-1}$ in the examples, and he was led to state the "tentative conjecture" that the results for the examples could be improved to agree with Roth's theorem. In this paper, the example in the two-dimensional case is studied and it is shown that at least for the two-dimensional case such an improvement is not possible.

Let $J(R, n)$ denote the set of R^n points of the form

$$\left(\frac{t_1}{R} + \frac{t_2}{R^2} + \cdots + \frac{t_n}{R^n}, \frac{t_n}{R} + \frac{t_{n-1}}{R^2} + \cdots + \frac{t_1}{R^n}\right)$$

where $t_i = 0, 1, 2, \cdots, R - 1$. We assume without loss of generality that the set of points is ordered so the x coordinates form an increasing sequence.

Points of the unit square which are of the form $(k/R^n, l/R^n)$, where k and l are positive integers, will be called "lattice points" of the unit square (with respect to $J(R, n)$). $T(R, n)$ will denote the average error of $J(R, n)$ at the lattice points, and it is shown in Section 3 that $T(R, n) = nT(R, 1) = n(R - 1)(R + 1)/12R$.

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