A TOPOLOGICAL H-COBORDISM THEOREM FOR $n \geq 5$

BY

E. H. CONNELL[1]

An H-cobordism is a compact manifold M with boundary components N and $\bar{N}$ which are deformation retracts of M. If $M = M^n$ is a simply connected differentiable manifold and $n \geq 6$, then M is diffeomorphic to $N \times I$ [11]. If M is a combinatorial manifold and $n \geq 5$, then $M - \bar{N}$ is piecewise-linearly homeomorphic to $N \times [0, 1)$ (p. 251 of [14]). In this paper it will be shown that if M is a topological n-manifold and $n \geq 5$, then $M - \bar{N}$ is homeomorphic to $N \times [0, 1)$. This is done by a type of topological engulfing (see Lemma 1).

A stronger form of Lemma 1 has independently (and previously) been obtained by M. H. A. Newman [1]. A corollary to these procedures is that if Y is a closed topological manifold which is a homotopy sphere, and $n \geq 5$, then Y is homeomorphic to S^n. The reader is assumed familiar with the proof of the combinatorial engulfing lemma [2], [5], [8].

Notation. Suppose M is a metric space with the distance between x and $y \in M$ denoted by $d(x, y)$. If $Y \subset M$ is any subset of M, $d(x, Y)$ will denote the distance from x to Y, $d(Y)$ will denote the diameter of Y, and for any $\varepsilon > 0$, $V(Y, M, \varepsilon)$ will denote the set $\{z \in N : d(z, Y) < \varepsilon\}$. If K is a finite complex, the statement that $f : K \to R^n$ is piecewise-linear (p.w.l.) means $\exists$ a subdivision K_1 of K such that any simplex σ of K_1 is mapped linearly into R^n by f. If M is a topological manifold, the interior and boundary of M are denoted by Int M and ∂M respectively. D^n denotes the closed n-cell in R^n,

$$D^n = \{(x_1, x_2, \cdots x_n) : -1 \leq x_i \leq 1, i = 1, 2, \cdots n\}.$$

Hypothesis I. $M = M^n$ is a compact, connected topological n-manifold ($n \geq 5$) with boundary consisting of two components, $\partial M = N \cup \bar{N}$; $\pi_i(M, N) = \pi_i(M, \bar{N}) = 0$ for $i = 1, 2, \cdots, n - 3$;

$$g : N \times [0, 1] \to M - \bar{N} \quad \text{and} \quad \bar{g} : \bar{N} \times [0, 1] \to M - N$$

are topological embeddings with $g(x, 0) = x$ for all $x \in N$ and $\bar{g}(y, 0) = y$ for all $y \in \bar{N}$. (Note: If M is any topological manifold with boundary components N and $\bar{N}$, then it follows from [13] that the embeddings g and $\bar{g}$ exist.)

LEMMA 1. *Suppose Hypothesis I. Suppose $K \subset R^n$ is a finite m-complex (a rectilinear complex in R^n), $m \leq n - 3$, $h : R^n \to$ Int M is a topological embedding, and ε is a number with $0 < \varepsilon < 1$. Then $\exists$ a homeomorphism*

Received April 25, 1966.

[1] The author has been supported by the Sloan Foundation and by a National Science Foundation grant.