THE BAR CONSTRUCTION AND ABELIAN \( H \)-SPACES

BY

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If \( X \) is an associative \( H \)-space with unit Dold and Lashof [2] have given a method for constructing a classifying space \( B_X \) which generalizes the classifying spaces for topological groups. In this note we present a construction which has three advantages over that of [2]:

1. if \( X \) is abelian \( B_X \) is also an abelian associative \( H \)-space with unit (Section 1),
2. if \( X \) is a CW complex and the multiplication is a cellular map then \( B_X \) is also a CW complex and the cellular chain complex of \( B_X \) is isomorphic to the bar construction on the cellular chain complex of \( X \) (Section 2),
3. there is an explicit diagonal approximation \( D : B_X \rightarrow B_X \times B_X \) which is cellular and in the cell chain complex of \( B_X \) induces exactly Cartan’s diagonal approximation for the bar construction (Section 3).

These properties are all easily established and once obtained are applied in Section 4 to give elementary constructions for the Eilenberg-Maclane spaces and to deduce the algebraic and geometric preliminaries to Cartan’s calculations of \( H^*(K(\pi, n)) \).

Remark. The recent results of J. C. Moore and S. Eilenberg [1] and N. Steenrod and E. Rothenberg [8] which are applied to study \( H^*(B_X) \) from information on the homology algebra \( H_*(X) \) may also be easily developed using our techniques.

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1. The construction

In this section we define the \( i \)-classifying spaces \( B^i(X) \) for a given associative \( H \)-space \( X \) with identity \( * \) and prove some of their more important properties. Most of these results are well known in one form or another [2], [5], [6], the only novel results being 1.6, 1.7, which exhibit the abelian multiplication in \( B_X \) and play a vital role in the applications.

Let \( \sigma^n \) be the Euclidian \( n \)-simplex represented as the set of points \((t_1, \cdots, t_n)\) in \( \mathbb{R}^n \) with

\[ 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1. \]

It has faces \( \sigma_i^n \) (for which \( t_i = t_{i+1} \) \( 1 \leq i < n \), \( \sigma_0^n \) \((t_1 = 0)\), and \( \sigma_n^n \) \((t_n = 1)\).

Let \( A^i(X) \), \( 0 \leq i \leq \infty \), be the disjoint union \( \sum_{j=0}^i X \times \sigma^j \times X^j \), (\( X^j \) is the \( j \)-fold Cartesian product). In \( A^i \) generate an equivalence relation by

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