THE BAR CONSTRUCTION AND ABELIAN H-SPACES

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If X is an associative H-space with unit Dold and Lashof [2] have given a method for constructing a classifying space B_X which generalizes the classifying spaces for topological groups. In this note we present a construction which has three advantages over that of [2]:

(1) if X is abelian B_X is also an abelian associative H-space with unit (Section 1),

(2) if X is a CW complex and the multiplication is a cellular map then B_X is also a CW complex and the cellular chain complex of B_X is isomorphic to the bar construction on the cellular chain complex of X (Section 2),

(3) there is an explicit diagonal approximation $D : B_X \to B_X \times B_X$ which is cellular and in the cell chain complex of B_X induces exactly Cartan's diagonal approximation for the bar construction (Section 3).

These properties are all easily established and once obtained are applied in Section 4 to give elementary constructions for the Eilenberg-Maclane spaces and to deduce the algebraic and geometric preliminaries to Cartan's calculations of $H^*(K(\pi, n))$.

Remark. The recent results of J. C. Moore and S. Eilenberg [1] and N. Steenrod and E. Rothenberg [8] which are applied to study $H^*(B_X)$ from information on the homology algebra $H_*(X)$ may also be easily developed using our techniques.

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1. The construction

In this section we define the i-classifying spaces $B^i(X)$ for a given associative H-space X with identity $*$ and prove some of their more important properties. Most of these results are well known in one form or another [2], [5], [6], the only novel results being 1.6, 1.7, which exhibit the abelian multiplication in B_X and play a vital role in the applications.

Let σ^n be the Euclidian n-simplex represented as the set of points $(t_1, \cdots, t_n)$ in R^n with

$$0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1.$$

It has faces σ_i^n (for which $t_i = t_{i+1}$) $1 \leq i < n$, σ_0^n $(t_1 = 0)$, and σ_n^n $(t_n = 1)$.

Let $A^i(X)$, $0 \leq i \leq \infty$, be the disjoint union $\sum_{j=0}^{i} X \times \sigma^j \times X^j$, ($X^j$ is the j-fold Cartesian product). In A^i generate an equivalence relation by

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