ON THE PROPAGATOR EQUATION

BY
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1. We are concerned here with weak and strong solutions of the evolution equation

$$(1.1) \qquad y' + A(t)y = f(t); \qquad y(\tau) = y_0$$

and related abstract equations. A number of observations and theorems will be given with no particular attempt at a "unified theory" (see [15] for a more complete discussion). Thus Part 2 is on strong solutions and propagators $G(t, s)$ solving (1.1) in the form

$$(1.2) \qquad y = G(t, \tau)y_0 + \int_\tau^t G(t, s)f(s)\, ds,$$

Part 3 contains some new results on weak solutions, and Part 4 is on some abstract problems. Some of the results have been announced in [16].

2. We suppose $A(t)$ is an unbounded linear operator in the separable Hilbert space H with domain $D_t = D(A(t))$ usually dense but this will be specified in each case. To begin with we suppose the problem (1.1) can be solved (uniquely) for

$$y_0 \in I_\tau \subset H \quad \text{and} \quad f(\cdot) \in F_\tau \subset L^2(H) = E$$

for some linear spaces I_τ and F_τ; furthermore we will deal with the finite interval case $\tau \leq t \leq T < \infty$ in general since all of the main features of the problem are exhibited there. We stipulate that all derivatives are in $D'(H)$ (see [33]) and the terms in (1.1) are in $L^2(H)$. First we note a somewhat stronger form of a lemma proved in [10] which is surely well known but seems not to have been written down in this form. Let $A(t)$ be accretive i.e., $\operatorname{Re}(A(t)x, x) \geq 0$, and let y be a unique solution of (1.1) with $y_0 = 0$ which we write $y = K(f)$. Now

$$(2.1) \qquad \begin{aligned} \operatorname{Re}(f, K(f))_E &= \operatorname{Re} \int_\tau^T (y' + Ay, y)_H\, dt \\ &= \frac{1}{2} \int_\tau^T \frac{d}{dt} \| y \|^2\, dt + \operatorname{Re} \int_\tau^T (Ay, y)\, dt \geq 0 \end{aligned}$$

(see [3] for integration theory in $L^2(H)$). If F_τ is dense and K extends by continuity to a continuous map $\bar{K} : E \to E$ then (2.1) extends to all E. But from (2.1) we can deduce also that

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