NONEXPANSIVE MAPPINGS AND FIXED-POINTS IN BANACH SPACES[1]

BY

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A subset K of a Banach space B has *normal structure* [2] if for each bounded convex subset H of K which contains more than one point there is a point $x \in H$ which is not a diametral point of H (that is, $\sup \{\| x - y \| : y \in H\} < \delta(H)$). We proved in an earlier paper [1] that if K is a bounded, nonempty, weakly compact, convex subset of a Banach space B, and if K has normal structure then every *finite* family $\mathfrak{F}$ of commuting nonexpansive mappings of K into itself has a common fixed-point. (A mapping f on K is nonexpansive if $\| f(x) - f(y) \| \leqq \| x - y \|$ for each $x, y \in K$.) If the norm of B is strictly convex then this theorem holds for infinite families. (For if the norm is strictly convex then the fixed-point set for each $f \in \mathfrak{F}$ is nonempty, bounded, closed and convex. Hence these fixed-point sets are weakly compact and have the finite intersection property; thus there is a point common to all of them.)

Although we do not know whether this theorem is true in general for infinite families, we show in this paper that by appropriately strengthening the condition of normal structure we are able to establish the existence of a common fixed-point for arbitrary families *without assuming strict convexity of the norm.* After proving a consequence of this, an observation is made about characterizations of Hilbert space due to Klee [6] and Phelps [7]. Finally, we show that the stronger version of normal structure introduced in this paper holds in compact convex sets and in closed convex subsets of uniformly convex Banach spaces.

1. Notation and definitions

Throughout the paper, the symbol $\delta(A)$ will denote the diameter of A, that is, $\delta(A) = \sup \{\| x - y \| : x, y \in A\}$, and $\overline{\text{co}}\, A$ will denote the closed convex hull of A. For $x \in B$, $\mathfrak{U}(x; r)$ and $\overline{\mathfrak{U}}(x; r)$ will denote, respectively, the open and closed spherical ball centered at x with radius r.

For subsets H and K of B, H bounded, let

$$r_x(H) = \sup \{\| x - y \| : y \in H\},$$

$$r(H, K) = \inf \{r_x(H) : x \in K\},$$

$$\mathcal{C}(H, K) = \{x \in K : r_x(H) = r(H, K)\}.$$

The set $\mathcal{C}(H, B)$ is frequently referred to as the Chebyshev center of H in B.

Received July 11, 1966.

[1] Research supported by a National Science Foundation grant.