MEASURES ON NON-SEPARABLE METRIC SPACES

BY

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1. Introduction

The main purpose of this note is to give a simpler and more general definition of "weak" or "weak-star" convergence of certain measures on non-separable metric spaces, and to prove its equivalence with the convergence introduced in [1] for the cases considered there.

Let (S, d) be a metric space. Let $\mathcal{B}$ or $\mathcal{B}(S)$ be the class of all Borel sets in S, i.e. the smallest σ-algebra containing all the open sets. One can safely assume that a finite, countably additive measure on $\mathcal{B}$ is concentrated in a separable subset [2]. It has seemed useful to consider finite, countably additive measures on metric spaces, not concentrated in separable subsets, defined on some, but not all, Borel sets [1]. Specifically, one can use the σ-algebra $\mathcal{U}$ or $\mathcal{U}(S)$ generated by the open balls

$$B(x, \varepsilon) = \{y \in S : d(x, y) < \varepsilon\}$$

for arbitrary x in S and $\varepsilon > 0$. Examples of finite measures on $\mathcal{U}$ not concentrated in separable subsets are the probability distributions of distribution functions of "empirical measures" [1]. For a simpler example, let S be uncountable and $d(x, y) = 1$ for $x \neq y$. Then $\mathcal{U}$ consists of countable sets, which we give measure 0, and sets with countable complement, which we give measure 1.

If S is separable, then all open sets are in $\mathcal{U}$ by the Lindelöf theorem, hence $\mathcal{U} = \mathcal{B}$. I don't know whether $\mathcal{U}$ is always strictly included in $\mathcal{B}$ for S non-separable, but it is in the cases mentioned above, and under the following conditions:

PROPOSITION. *Suppose that the smallest cardinal of a dense set in S is c (cardinal of the continuum). Then $\mathcal{U}$ has cardinal c and $\mathcal{B}$ has cardinal 2^c. Hence $\mathcal{U}$ is strictly included in $\mathcal{B}$.*

Proof. Let A be a dense set in S of cardinal c. Let G be the class of balls $B(x, r)$ with x in A and r (positive) rational. We show that G generates $\mathcal{U}$. Let $x \in S$, $r > 0$. Let $x_n \in A$, $x_n \to x$. We can assume $d(x_n, x) < r$ for all n. Let r_n be positive rational numbers such that $r_n \to r$ and $r_n < r - d(x_n, x)$ for all n. Then

$$B(x, r) = \bigcup_{n=1}^{\infty} B(x_n, r_n),$$

showing that G generates $\mathcal{U}$.

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