ON THE EQUATION $f_1 g_1 + f_2 g_2 = 1$ IN H^p.

BY

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1. Introduction and definition

Let D denote the unit disk in the complex plane and $\bar{D}$ its closure. We shall say that f is in H^p of the disk, $p \geq 1$, if f is holomorphic in D and satisfies

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} | f(re^{i\theta}) |^p \, d\theta < M < +\infty$$

for all $r < 1$. It is known that H^p is a complete normed linear space with

$$\| f \|_p = \lim_{r \to 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} | f(re^{i\theta}) |^p \, d\theta \right)^{1/p}.$$

In this paper we investigate the following equation

$$f_1(z)g_1(z) + f_2(z)g_2(z) = 1, \qquad z \in D \tag{1.1}$$

in the following sense. Given f_1 and f_2 in H^p and H^r respectively, what conditions are necessary to guarantee the existence of the pair g_1 and g_2 in some Hardy spaces satisfying (1.1). We show by examples one cannot always hope for solutions. We study the structure of the class of the given function pairs f_1 and f_2 and also the structure of the solution pairs g_1 and g_2.

Our study is motivated by the classical results of W. Rudin, D. J. Newman and L. Carleson. Since we use their results we state them here. Let H^∞ denote the space of bounded holomorphic functions in D with the sup norm. The closed subalgebra of H^∞ consisting of those functions which are also continuous on $\bar{D}$ is denoted by A (of $\bar{D}$). In [5] Rudin showed that if f_1 and f_2 are in A and $| f_1 | + | f_2 | > 0$ on $\bar{D}$ then the ideal generated by f_1 and f_2 is A, or there exist solutions g_1 and g_2 in A satisfying (1.1) on $\bar{D}$. Moreover, D. J. Newman has indicated that proving for f_1 and f_2 in H^∞ with $| f_1 | + | f_2 | \geq \delta > 0$ we can find g_1 and g_2 in H^∞ satisfying (1.1) on D is equivalent to showing that the point evaluations on D are dense in the maximal ideal space of H^∞. Carleson's [1] solution of this (Corona) problem has completed the H^∞ phrase of the problem.

We wish to make the following convention. If $S = \{z; | z - z_0 | < \rho\}$ is a disk then A of $\bar{S}$ means those functions continuous in $\bar{S}$ and holomorphic in S.

2. The basic solution

The following result is known but we have not found a proof in the literature, therefore we include our proof not only for completeness but also because it gives us valuable information about the pairs of solutions of (1.1).

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