LUSTERNIK-SCHNIRELMANN CATEGORY AND STRONG CATEGORY

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1. Introduction

The purpose of this note is to compare the following two numerical homotopy invariants of a topological space.

DEFINITION 1.1 The Lusternik-Schnirelmann category, cat B, of a topological space B is the least integer $k \geq 0$ with the property that B may be covered by $k + 1$ open subsets which are contractible in B; if no such integer exists, cat $B = \infty$.

DEFINITION 1.2. The strong category, Cat B, of a topological space B is the least integer $k \geq 0$ with the property that B has the homotopy type of a CW-complex which may be covered by $k + 1$ self-contractible subcomplexes; if no such integer exists, Cat $B = \infty$.

The first definition is classical; the second is the homotopy invariant version of an earlier definition due to Fox [3, §IV] and was introduced in [4]. Since a CW-pair has the homotopy extension property and since a CW-complex is locally contractible, the CW-complex, say B', described in 1.2 satisfies cat $B' \leq k$. Therefore, and since category is a homotopy type invariant, one has cat $B \leq$ Cat B for any space B; in particular, Cat $B = \infty$ if B fails to have the homotopy type of a CW-complex. Our main result is expressed by

THEOREM 1.3. *Let B be an $(n - 1)$-connected CW-complex with* cat $B \leq k$ $(k \geq 1, n \geq 2)$. *If* $\dim B \leq (k + 2)n - 3$, *then also* Cat $B \leq k$.

It is well known that cat $B \leq 1$ if and only if B is an H'-space, and it follows from 2.1 below that Cat $B \leq 1$ if and only if B has the homotopy type of a suspension. Hence, 1.3 may be considered as a generalization of the following result: *any $(n - 1)$-connected H'-space B of dimension $\leq 3n - 3$ has the homotopy type of a suspension.* Under the additional assumption that the homology of B is finitely generated, this last result was first proved in [1], and an example therein reveals that 1.3 yields the best possible result at least when $k = 1$. The proof to follow is essentially different from that given in [1]. In the final section, we show that our approach leads to a substantial simplification of the main geometric result in [6] which relates category to the differentials in certain spectral sequences.

The preceding two definitions, as stated in terms of coverings by certain subsets, do not dualize in the sense of [2]. Nevertheless, it is possible to dualize the main results of the paper. Thus, the dual of 2.2 below yields a satisfactory

Received June 1, 1966.

[1] This work was partially supported by the National Science Foundation.