QUADRATIC MAPS AND STABLE HOMOTOPY GROUPS OF SPHERES

BY

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The original proof [4] of the Bott periodicity theorem used Morse theory. Recent work on this theorem has, however, been algebraic in nature [1], [2], [3], [7]. The new proofs of the Bott periodicity theorem center around showing that a stable homotopy class can be represented by a specially simple sort of polynomial map. In [1] Atiyah and Bott ask whether it might be possible to use this approach on other homotopy problems. Is there, for example, some specially simple class of polynomial maps which carries the stable homotopy of spheres? As a possible first step towards selecting such a class we shall indicate that probably one wants to examine the properties of quadratic maps. In detail, we shall show that:

1. The stable J-homomorphism can be interpreted as an algebraic operation which converts a linear map into a quadratic map.
2. Any element of a k-stem can be represented by a quadratic map $q : R^n \to R^l$ such that $q(S^{n-1}) \subset R^l - \{0\}$.

In view of these results it is not surprising that many classical examples of non-trivial maps from S^n to S^k are quadratic. The Hopf map $S^3 \to S^2$ is, for instance, given by

$$(x_1, x_2, x_3, x_4) \to (2x_1 x_3 - 2x_2 x_4, 2x_1 x_4 + 2x_2 x_3, -x_1^2 - x_2^2 + x_3^2 + x_4^2).$$

We remark that the results given here are mainly suggestive. No actual computations of k-stems are done. But perhaps eventually the k-stem may be envisaged as a group of equivalence classes of quadratic forms.

Our main technical lemma is 2.9. The proof of this lemma describes a procedure for lowering the degree of a polynomial map. This procedure resembles the linearization procedure of [1].

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R. Wood has also, independently, proved that any element in a k-stem can be represented by a quadratic map.

1. The J-Homomorphism

1.1. *Notation.* R = the real numbers.

R^n = the space of all n-tuples a, $a = (a_1, \cdots, a_n) a_i \in R$, $\| a \| = (a_1^2 + \cdots + a_n^2)^{1/2}$.

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