A LINEAR EXTENSION THEOREM

BY

E. MICHAEL[1] AND A. PEŁCZYŃSKI[2]

1. Introduction

Let T be a topological space, S a closed subset of T, and $C(S)$ and $C(T)$ the Banach spaces of bounded, continuous complex (or real) functions on S and T, respectively. Let $E \subset C(S)$ and $H \subset C(T)$ be closed subspaces. A continuous linear map $u : E \to H$ is called a *linear extension* if $u(f)$ is an extension of f for every $f \in E$. The purpose of this paper is to study the existence of linear extensions of norm one.

If $H = C(T)$, our problem was completely settled by Borsuk [3] for separable metric T, and subsequently by Dugundji [6, Theorem 5] for all metric T.[3]

THEOREM 1.1 (Borsuk-Dugundji). *If T is metrizable, there exists a linear extension $u : C(S) \to C(T)$ of norm one.*

If H is a proper subspace of $C(T)$, the situation becomes more complicated, and Example 9.2 shows that no linear extension $u : C(S) \to H$ need exist even when every $f \in C(S)$ can be extended to some $f' \in H$. We therefore introduce the following concept:

DEFINITION 1.2. The pair (E, H) has the *bounded extension property* if, given any $\varepsilon > 0$, every $f \in E$ has a bounded family of extensions

$$\{f_{\varepsilon,W} : W \supset S, W \text{ open in } T\} \subset H$$

such that $|f_{\varepsilon,W}(x)| \leq \epsilon$ whenever $x \in T - W$.

Note that the pair $(C(S), C(T))$ has this property whenever T is normal. The following result was proved by the second author in [13] and [14].[4]

THEOREM 1.3. *If T is compact metric, and if $(C(S), H)$ has the bounded extension property, then there exists a linear extension $u : C(S) \to H$ of norm one.*

Perhaps the most interesting application of Theorem 1.3 was to the case where T is the unit circle in the complex plane, $H \subset C(T)$ is the disc algebra (i.e. H consists of boundary values of continuous functions on the unit disc

Received July 18, 1965.

[1] Partially supported by a National Science Foundation grant.

[2] Partially supported by a National Science Foundation grant.

[3] Strictly speaking, Borsuk and Dugundji stated the theorem for real scalars, but their proofs remain valid for complex scalars as well (which means, in particular, that u is then complex-linear).

[4] To be precise, [13] and [14] assume a property which is formally stronger than the bounded extension property, but which (see Corollary 5.3) is actually equivalent to it.