TRIVIAL LOOPS IN HOMOTOPY 3-SPHERES

BY

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In this paper we show that every homotopy 3-sphere possesses a cell-decomposition $\Gamma$ which is in some respect especially simple:

**Theorem.** If $M^3$ is a homotopy 3-sphere then there exists a cell-decomposition $\Gamma$ of $M^3$ with the following properties:

(i) $\Gamma$ consists of one vertex $E^0$, $r$ open 1-cells, $E^1_1, \ldots, E^1_r$, $r$ open 2-cells, $E^2_1, \ldots, E^2_r$, and one open 3-cell $E^3$.

(ii) There exist (nonsingular, polyhedral) disks $V^2_1, \ldots, V^2_r$ in $M^3$ such that $E^i_i \cap V^2_i$ for all $i = 1, \ldots, r$.

(iii) The disks $V^2_1, \ldots, V^2_r$ may be chosen such that the connected components of $V^1_i \cap V^2_j - E^0$ $(i \neq j, \text{between } 1 \text{ and } r)$ are normal double arcs in which $V^1_i$ and $V^2_j$ pierce each other such that the interior of each double arc lies in $E^1_i \cap V^2_j$, one of its boundary points lies in $E^1_i$, and the other one lies in $E^1_j$ (see Fig. 1), and such that $V^2_i \cap V^2_j \cap V^2_k = E^0$ (if $i, j, k$ are pairwise different, between 1 and $r$).

It is a known fact that every closed 3-manifold $M^3$ possesses a cell-decomposition $\Gamma$ with property (i) (this follows easily from results in Seifert-Threlfall [4], see [2, Sec. 5]). If $M^3$ is a homotopy 3-sphere, i.e., simply connected, then this is equivalent to the fact that the 1-skeleton $G^1 = \bigcup_{i=1}^{r} E^1_i$ of $\Gamma$ bounds a "singular fan" in $M^3$ (see [2, Sec. 6]). Now property (ii) of $\Gamma$ means that $G^1$ is a wedge of trivial loops in $M^3$, and (iii) means that $G^1$ bounds a singular fan $\bigcup_{i=1}^{r} V^2_i$, which is especially simple in the sense that its single leaves $V^2_i$ are nonsingular.

As Bing has shown in [1] it would be sufficient for a proof of the Poincaré conjecture if one could show that every polyhedral, simple closed curve in $M^3$ lies in a 3-cell in $M^3$, or that the 1-skeleton $G^1$ of some cell-decomposition $\Gamma$ of $M^3$ lies in a 3-cell in $M^3$. The property (ii) of $\Gamma$ means that every single closed curve $E^1_i \subset G^1$ lies not only in a 3-cell $V^2_i$ (which may be obtained as a small neighborhood of $V^2_i$) in $M^3$ but moreover is unknotted in that 3-cell $V^2_i$. So one may hope that the above theorem could be used as a tool for proving the Poincaré conjecture or for deriving further partial results on homotopy 3-spheres.

**Proof of the theorem**

1. **Preliminaries.** We choose the semilinear standpoint as described in [3, Sec. 3], i.e., we assume for convenience that $M^3$ is a piecewise rectilinear