ALGEBRAICALLY TRIVIAL DECOMPOSITIONS OF HOMOTOPY 3-SPHERES[1]

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Every compact 3-manifold M^3 without boundary possesses a cell-decomposition Ψ that contains just one vertex, say O, (see for instance [3, Sec. 5]). From Ψ we may read by a well-known procedure (see [7, §62]) a *"corresponding" presentation*

$$\mathfrak{P}(\Psi) = (\{g_1, \cdots, g_a\}, \{\mathfrak{r}_1, \cdots, \mathfrak{r}_b\})$$

of the fundamental group $\pi_1(M^3)$ where the generators $g_1, \cdots, g_a$ are in 1-1 correspondence with the (oriented) 1-dimensional elements $E_1^1, \cdots, E_a^1$ of Ψ and the relators $\mathfrak{r}_1, \cdots, \mathfrak{r}_b$ are in 1-1 correspondence with the 2-dimensional elements $E_1^2, \cdots, E_b^2$ of Ψ, i.e., $\mathfrak{r}_j$ is a word in the $g_i^{\pm 1}$'s obtained by running once around the boundary of E_j^2. In this way the relators $\mathfrak{r}_j$ are uniquely defined up to cyclic permutations and inversions, i.e., if we denote by $\langle \mathfrak{r}_j \rangle$ the set of all cyclic permutations of $\mathfrak{r}_j$ and of $\mathfrak{r}_j^{-1}$ then the $\langle \mathfrak{r}_j \rangle$'s are uniquely defined.

In the special case that M^3 is a homotopy 3-sphere, $\mathfrak{P}(\Psi)$ is a presentation of the trivial group. However, it is—in general—an unsolved problem to recognize whether or not a given presentation $\mathfrak{P}(\Psi)$ presents the trivial group; this problem seems to be extremely difficult and it may be unsolvaable, since the triviality problem of group theory is unsolvable (see [1], [6]). One might expect that these group theoretic difficulties are also the reason for the difficulties of the Poincaré problem. But the result of this paper shows that this is not so: We shall prove that every homotopy 3-sphere M^3 possesses a cell-decomposition Ψ such that the corresponding presentation

$$\mathfrak{P}(\Psi) = (\{g_1, \cdots, g_a\}, \{\mathfrak{r}_1, \cdots, \mathfrak{r}_b\})$$

is *obviously trivial*, i.e., such that $\mathfrak{P}(\Psi)$ can be transformed by simple cancellation operations (without changing the generators g_i and the number b of relators) into the *"standard trivial presentation"*

$$\mathfrak{O} = (\{g_1, \cdots, g_a\}, \{g_1, \cdots, g_a, \overset{b-a}{*}\})$$

where $\overset{b-a}{*}$ means that $\mathfrak{O}$ contains $b - a$ times the empty relator (i.e., the relations of $\mathfrak{O}$ are[2] $g_1 = 1, \cdots, g_a = 1$, and $b - a$ times the trivial relation $1 = 1$). To make this precise we say that a presentation $\mathfrak{P}''$ is obtained from

Received January 16, 1967.

[1] This research was supported by the Air Force Office of Scientific Research and by the National Science Foundation.

[2] Here the equality sign means that both sides of the equation represent the same group element; but in general, if not stated otherwise, we call two words equal if and only if they read, letter by letter, in the same way.