SUBGROUP-DETERMINING FUNCTIONS ON GROUPS

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I. Introduction and notation

Let G be a group and let S be a subset of G. Suppose that S is a subgroup of G if and only if for every $x, y \in S$, $f(x, y) \in S$. What forms may the function f take?

This question was first raised by Higman and Neumann [6] and investigated by Hulanicki and Świerczkowski [8], who introduced the following definition:

DEFINITION. A group G has *property P* if and only if there exist integers $a_i, b_i, i = 1, \cdots, r$, and $m_j, n_j, j = 1, \cdots, s$, such that

(i) the word

$$x \circ y = x^{a_1}y^{b_1} \cdots x^{a_r}y^{b_r} \tag{1}$$

defines a binary operation in G, not identically equal in G to xy or to yx;

(ii) the elements of G form a group $G_\circ$ under the operation $x \circ y$, in which the m^{th} power of x is denoted by $[x]_\circ^m$, the inverse of x by $x^{[-1]}$ and the commutator of y and x by $[y, x]_\circ$;

(iii) the operation xy is a word in $G_\circ$, i.e. the law

$$xy = [x]_\circ^{m_1} \circ [y]_\circ^{n_1} \circ \cdots \circ [x]_\circ^{m_s} \circ [y]_\circ^{n_s} \tag{2}$$

holds identically for every $x, y \in G$.

In this case, $x \circ y$ is called an *s-function* on G.

They pointed out that, if G has property P, then $x \circ y^{[-1]}$ is a subgroup-determining function on G, different from the obvious ones, namely $f_1(x, y) = xy^{-1}$, $f_2(x, y) = x^{-1}y$ and their transposes.

It follows from results in [6], [10] and [16] that neither an Abelian nor a free group possesses property P, and that no s-function may be defined on the variety of all groups, nor on the class of all finite nilpotent groups, nor on the class of all finite p-groups, for p a given prime. However, in [8] it is shown that if G is nilpotent of class 2 and if its commutator subgroup, G', has finite exponent, then G has property P, and all possible s-functions on such a group are determined. G and $G_\circ$ are shown to be isomorphic if G is also periodic.

In this paper, we discuss further classes of groups with property P, the s-functions that can be defined on them and the relation between G and $G_\circ$. In II, we prove the following:

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