A CLASS OF REPRODUCING KERNELS[1]

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Let $P(x)$ be a homogeneous positive definite polynomial of order $2m$, $m > 0$ an integer, in $x \in E_n$ having constant coefficients. We will show that the kernel

$$K(x, y) = \int_{z \in E_n} e^{-2\pi i x \cdot z} \exp\left(2\pi i\beta \mid P(z) \mid^{1/2m} y\right) dz,$$

where $x \in E_n$, $y > 0$, $\beta^{2m} + 1 = 0$, the imaginary part of β is positive and the real part of β^2 is negative, satisfies the following five properties:

(1) $K(x, y) \in L^1(E_n)$, independently of y;

(2) $\int_{E_n} K(x, y)\, dx = 1$;

(3) $\int_{|x| \geq \delta > 0} \mid K(x, y) \mid^q dx \to 0$ as $y \to 0$, $1 \leq q$;

(4) $\mid K(x, y) \mid < Ay^{-n}$, A independent of x, y;

(5) $\mid K(x, y) \mid < By \mid x \mid^{-n-1}$, B independent of x, y.[2]

These are sufficient to guarantee that K is a reproducing kernel in the sense that, if we define

$$f(x, y) = f * K(x, y) = \int_{z \in E_n} f(z) K(x - z, y)\, dz$$

then $f(x, y) \to f(x)$ as $y \to 0$ in L^p norm and almost everywhere for any $f \in L^p(E_n)$, $1 \leq p < \infty$.

These kernels are of interest, since $K(x, y)$ and, hence, $f(x, y)$, as defined above will satisfy the elliptic equation

$$(\partial^{2m}/\partial y^{2m})u + P(D)u = 0$$

in $E_{n+1}^+ = \{(x, y) \mid x \in E_n, y > 0\}$, where $P(D)$ is the differential operator obtained from $P(x)$ by replacing each occurrence of x_i by $\partial/\partial x_i$, $i = 1, \cdots, n$.

Letting $x = \mid x \mid x'$, $\mid x \mid = \sum_{i=1}^n x_i^2$, from the homogeneity of P, we obtain by a simple change of variable the following identities for K:

$$K(x, y) = y^{-n} K(xy^{-1}, 1) = \mid x \mid^{-n} K(x', y \mid x \mid^{-1}) \quad \text{for all } x \in E_n,\ y > 0.$$

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[2] Throughout this paper the letters A, B, C will denote constants which are independent of x, y, z, t, or v. They may, however, have differing values in different parts of the same argument.