ASYMPTOTIC EXPANSIONS FOR THE COEFFICIENTS OF ANALYTIC FUNCTIONS[1]

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1. Introduction

The problem of obtaining an asymptotic expansion for the coefficient α_n of $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ arises in many instances. The chief sources of such problems are number theory and the general area of combinatorics.

There are various aspects of this problem depending on how much precision one requires. In the simplest case, one asks only for an asymptotic formula of the kind $\alpha_n \sim c_n$, as $n \to \infty$, where c_n is a relatively simple function of n. On the other hand, one may require that a full asymptotic expansion of the type

$$(1) \qquad \alpha_n = c_n\{1 + F_1(n)/\beta_n + \cdots + F_N(n)/\beta_n^N + o(F_N(n)/\beta_n^N)\}$$

hold for each $N \geqq 0$ as $n \to \infty$; here, one allows $F_k(n)$ to depend on n but wishes to have

$$F_{k+1}(n)/\beta_n^{k+1} = o(F_k(n)/\beta_n^k)$$

for each k, as $n \to \infty$, and one also desires that $\beta_n \to \infty$ as $n \to \infty$. In fact, in most cases, $F_k(n) = o(\beta_n^\varepsilon)$ for each $\varepsilon > 0$ and each k, as $n \to \infty$.

The literature contains many papers dealing with such problems. Here we mention only the papers of Hayman [4], Grosswald [1], and a previous paper [2] of ours. Hayman deals with the simple formula $\alpha_n \sim c_n$ under relatively weak conditions on $f(z)$. Grosswald, however, assumes more and obtains a result of the type (1). In our earlier paper, we also obtained such a result but for the special function $f_0(z) = \exp(ze^z)$ which does not satisfy Grosswald's hypotheses. In the present work, we generalize our earlier theorem by using some ideas in Grosswald's and Hayman's papers; this yields a result having weaker hypotheses than Grosswald's. Finally, we apply our theorem to $f_0(z)$ which is the exponential generating function of U_n, the number of idempotent elements in the symmetric semigroup on n letters.

2. Statement of the result

We make the following assumptions (A)–(E):

(A) $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ is analytic for $|z| < R$, $0 < R \leqq \infty$, and is real for real z.

(B) There exists an $R_0 \in (0, R)$ and a $d(r)$ defined for all $r \in (R_0, R)$ such

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