A MONOTONIC MAPPING THEOREM FOR SIMPLY CONNECTED 3-MANIFOLDS[1]

BY

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1. Statement of results

THEOREM. *Let M be a triangulated 3-manifold, and suppose that M is compact, connected and simply connected. Then there is a subcomplex K of a triangulation of the 3-sphere S^3, and a mapping*

$$f : S^3 \to M$$

of S^3 onto M, such that

(1) $\dim K \leqq 2$,

(2) *$f \mid K$ is simplicial (relative to K and a subdivision of M),*

(3) *$f \mid (S^3 - K)$ is one-to-one,*

(4) $f(K) \cap f(S^3 - K) = 0$,

(5) *f is monotonic, and*

(6) *Each set $f^{-1}(x)$ is either a point or a linear graph.*

Here (5) means that each set $f^{-1}(x)$ is connected. By a linear graph we mean a 1-dimensional polyhedron.[2]

2. Bing's example

R. H. Bing [B] has given a curious example of a mapping of the sort described in the above theorem. In Bing's example, M is S^3, but the inverse-image sets $f^{-1}(x)$ are of an unexpected sort. Consider (as shown on the left in Figure 1) two circular disks D_1, D_2 which intersect each other in a common radius. Let their boundaries be the circles C_1 and C_2. Each of these is decomposed into concentric circles. (In the figure, we show one such circle J_1 in D_1, and one such circle J_2 in D_2.) Thus we have a collection G of sets, consisting of (1) the points of $S^3 - (D_1 \cup D_2)$, (2) the circles C_1 and C_2 and (3) infinitely many "figure 8's" of the type $J_1 \cup J_2$.

The collection G is upper-semicontinuous in the usual sense: if X is any closed set in S^3, then the union of all elements of G that intersect X is also a closed set [K]. Thus we can define a Hausdorff topology in G, by saying

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[1] A portion of the work reported here (in Sections 3 through 10 below) was done while the author held a Guggenheim Memorial Foundation Fellowship. This portion of the paper was also sponsored by the National Science Foundation and the Institute for Advanced Study.

[2] Theorem 3.1 below was announced in [M] (see the bibliography at the end), and earlier, in colloquia at Warsaw and Madison. Since then, a weaker version of the theorem has been proved by Wolfgang Haken [H_1].