HARMONIC FUNCTIONS ON THE UNIT DISC I[1]

BY

GUY JOHNSON, JR.

1. Introduction

This report concerns real- or complex-valued harmonic functions defined on discs in the plane. The principal result may be stated as

THEOREM A. *A function f is harmonic on the unit disc if and only if there is a sequence $\{g_n\}$ of continuous functions on the unit circle such that*

$$\lim (n! \, \| g_n \|)^{1/n} = 0 \tag{1.1}$$

and

$$f(r, \theta) = \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} P_r^{(n)}(\theta - t) g_n(t) \, dt. \tag{1.2}$$

In (1.1) the norm is defined by $\| g \| = \sup \{ | g(t) | : 0 \leqq t \leqq 2\pi$ and in (1.2)

$$P_r(\theta - t) = \mathcal{R}[(e^{it} + re^{i\theta})/(e^{it} - re^{i\theta})]$$

is the Poisson kernel for the unit disc. $P_r^{(n)}$ is the nth derivative of P_r.

This theorem was reported in [5]. It has been used by Douglas [2] as a global constraint for harmonic continuation in the disc of a function which is approximated at a finite set of points. Saylor, a student of Douglas, has extended these results to the case of solutions of a linear elliptic second order partial differential equation with analytic coefficients on a domain in R^n bounded by a compact analytic boundary [10].

There is a rich boundary-value theory concerning the Poisson and the Poisson-Stieltjes integrals beginning with the work of Fatou. A nice treatment of old and new results in this theory may be found in [4]. In view of the above result it is natural to ask whether there is a boundary function, in some generalized sense, associated with an arbitrary harmonic function by means of a Poisson representation. In fact, denoting $f_r(\theta) = f(r, \theta)$ and using (1.2) we have, formally

$$f_r = \sum_{n=0}^{\infty} P_r^{(n)} * g_n = \sum_{n=0}^{\infty} P_r * g_n^{(n)} = P_r * \sum_{n=0}^{\infty} g_n^{(n)} = P_r * g.$$

The appropriate setting in which these calculations have meaning is a generalized function space having analytic test functions. Let $\mathcal{H}$ denote the linear space of analytic complex-valued functions on the unit circle Γ. Köthe [14], [15] introduced a certain locally convex topology for $\mathcal{H}$. The strong dual $\mathcal{H}'$ is the space of generalized functions. The Fantappie indicator of an element $f \in \mathcal{H}'$ is a function holomorphic on $\Omega - \Gamma$ and zero at ∞. Ω denotes

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