THE DENSEST IRREGULAR PACKING OF THE MORDELL CUBIC NORM-DISTANCE

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1. Introduction

Let $\mathcal{S}$ be a star-domain, symmetric about 0. A set of points $\mathcal{P}$ is said to provide a packing for $\mathcal{S}$ if the domains $\{\mathcal{S} + \mathcal{P}\}$, where $P \in \mathcal{P}$, have the property that no domain $\mathcal{S} + P_0$ contains the center of another in its interior. We also say that $\mathcal{P}$ is an $\mathcal{S}$-admissible point set. A packing $\mathcal{P}$ is said to be regular if $\mathcal{P}$ is an $\mathcal{S}$-admissible lattice; it is said to be semi-regular if it is the union of a lattice $\mathcal{L}$ and a translate of $\mathcal{L}$; it is said to be irregular if it is not necessarily a lattice or a union of lattices.

The domain of action method developed by M. Rahman has been employed by Sister M. R. Von Wolff to determine that the densest irregular packing of the star-domain $\mathcal{S}_1 : |xy| \leqq 1$ has the density of an $\mathcal{S}_1$-critical lattice.

It is the purpose of this paper to exhibit further the strength of the domain of action method in the determination of the best possible irregular packing of non-convex regions. The method is applied to the star-domain $\mathcal{S}_2 : |y(3x^2 - y^2)| \leqq 1$ which is equivalent to the region

$$\mathcal{S}_2 : |x^3 - x^2y - 2xy^2 - y^3| \leqq 1$$

for which L. J. Mordell [3] has determined the critical lattices. R. P. Bambah [1] gave another proof of this result by determining the critical determinant and the two critical lattices of the region $\mathcal{S}_2$.

Consider the square $|x| < t$, $|y| < t$ Let $A(t)$ denote the number of points of a set $\mathcal{P}$ in the square; then the density of $\mathcal{P}$, denoted $\mathfrak{D}(\mathcal{P})$, is defined as $\limsup_{t\to\infty} A(t)/4t^2$.

From the definition it follows that for any two-dimensional lattice $\mathcal{L}$ the density $\mathfrak{D}(\mathcal{L})$ is the reciprocal of its mesh.

A norm-distance, [2, p. 103], is a real-valued function $n(X) = n(OX)$, defined on the plane, such that $n(X)$ is

(1) nonnegative; i.e., $n(X) \geqq 0$;
(2) continuous;
(3) homogeneous; i.e., $n(tX) = |t|\, n(X)$, where t is any real number.

A convex distance function or Minkowski distance, m, is a norm-distance with the additional properties:

(1) $m(PQ) = 0$ implies $P = Q$.
(2) $m(PQ) \leqq m(PR) + m(RQ)$.

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