SEPARABILITY OF TORSION FREE GROUPS AND A PROBLEM OF J. H. C. WHITEHEAD

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1. Introduction

Our investigation of locally free groups is motivated by a question posed by J. H. C. Whitehead which asks for a characterization of those groups G for which Ext $(G, Z) = 0$. Such a group G is called a Whitehead group or more simply a W-group. Stein [6], Rotman [5], Chase [1], [2] and Nunke [4] have investigated these groups and have established a number of conditions that are necessary in order that a group be a W-group. The most notable necessary conditions are that a W-group must be locally free, totally separable, slender and satisfy Rotman's density condition [5]. It is the purpose of this paper to consider separability conditions on a group G and to study their effect on the groups Ext (G, Z) and Ext (G, S) where $S = \sum_{\aleph_0} Z$. Specifically, we wish to find rather natural sufficient conditions on the group structure of a group G in order that G be a W-group. These conditions appear on the surface to be weaker than the obvious condition that G be free. In Section 3 we establish our most striking result which states that Ext $(G, S) = 0$ if and only if G is locally free and $\aleph_1$-coseparable (see definition below). We also show that if G is a locally free, totally $\aleph_1$-separable group, then Ext $(G, S) = 0$. Hence either of the above conditions is sufficient for G to be a W-group. Section 2 is devoted to characterizing locally free, coseparable groups as just those groups G such that Ext (G, Z) is torsion free. This result is essentially just a recasting of Chase's Theorem 4.2 [1] in terms of coseparability.

Throughout this paper all groups are abelian. For the most part, the terminology and notation is that of [3]. Let G be an $\aleph_1$-free group (i.e. all countable subgroups of G are free). G is called separable ($\aleph_1$-separable) if every finitely (countably) generated subgroup of G is contained in a finitely (countably) generated direct summand of G.[2] We call G coseparable ($\aleph_1$-coseparable) if every subgroup H of G with the property that G/H is finitely (countably) generated contains a direct summand K of G such that G/K is finitely (countably) generated. If every subgroup of G is separable ($\aleph_1$-separable), we call G totally separable ($\aleph_1$-separable). Following R. J. Nunke, G will be called locally free if G is both separable and $\aleph_1$-free. It

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[2] Observe that our definition of separability agrees with the definition of Fuchs [3] for $\aleph_1$-free groups.