RISES OF NONNEGATIVE SEMIMARTINGALES[1]

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A real-valued stochastic process $f_0, f_1, \cdots$ has a *rise* of size y if $\exists i, j$ with $i < j$ such that $f_j - f_i \geq y$. This note obtains sharp upper bounds to the probability of a rise of size y for certain natural classes of stochastic processes.

Let Θ be a class of probability measures on the real line. If, for every n, given any partial history $f_0, \cdots, f_n$, the conditional distribution θ of the increment $f_{n+1} - f_n$ is in Θ, then $\{f_j\}$ is a *Θ-process*. If, in addition, $f_0 \equiv x$, then $\{f_j\}$ is an *(x, Θ)-process*. One can think of an (x, Θ)-process as the successive fortunes of a gambler whose initial fortune is x, and who chooses his successive lotteries from Θ.

Let $\rho(x, y) = \rho(x, y, \Theta)$ be the least upper bound over all nonnegative (x, Θ)-processes (including not necessarily countably additive processes) to the probability that the process experiences a rise of size y. The determination of ρ can sometimes be reduced to solving a simpler problem, namely that of determining U, where $U(x, y) = U(x, y, \Theta)$ is the least upper bound over all nonnegative (x, Θ)-processes $\{f_j\}$ to the probability that there is j with $f_j \geq y$.

As will soon be evident, there are interesting Θ for which

$$(1) \qquad U(x - m, y - m) = \frac{U(x, y) - U(m, y)}{1 - U(m, y)},$$

whenever $0 < m < x$, and $m < y$.

Incidentally, for every Θ, the left side of (1) is majorized by the right side. This inequality is quite simple to establish and is analogous to Theorem 4.2.1, p. 64 in [2].

I do not investigate the regularity conditions that U perhaps automatically satisfies once it satisfies (1), but, at least in interesting examples,

(2) $U(x, y)$ is convex in x for $0 \leq x \leq y$,

and

(3) $U(x, y)$ is continuously differentiable in x and y for $0 \leq x \leq y$.

Let

$$(4) \qquad \lambda = \lambda(y) = \frac{\partial U}{\partial x}(0, y).$$

THEOREM 1. *If U satisfies* (1), (2) *and* (3), *then*

$$\rho(x, y) = 1 - e^{-\lambda x}.$$

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