ON THE ANTICENTER OF NILPOTENT GROUPS

BY

WOLFGANG P. KAPPE

The anticenter $AC(G)$ of a group, as defined by N. Levine [3] is the subgroup generated by the set RG of elements with trivial centralizer. Here an element x is said to have trivial centralizer if $\langle x, y\rangle$ is cyclic for all $y \in c_G\langle x\rangle$. Free groups and a class of groups investigated by Greendlinger [2] are examples of infinite groups where every element has trivial centralizer. In a finite p-group P we have $RP = P$ if and only if there is at most one subgroup of order p, i.e. P is cyclic or a generalized quaternion group. If G is any finite group it follows easily that $RG = G$ if and only if the Sylow subgroups are cyclic or generalized quaternion groups. These groups have been classified by Zassenhaus [6, Satz 7] and Suzuki [5, Theorem E]. Abelian groups with $RG \neq 1$ are easily determined:

THEOREM A [1, Theorem 3]. *Assume $G \neq 1$ is an abelian group. $RG \neq 1$ if and only if G is either torsion free of rank 1 or G is a torsion group and at least one of the Sylow subgroups has rank 1.*

In all cases mentioned so far the anticenter coincides with the set of elements with trivial centralizer. Little is known about the structure and embedding of $AC(G)$ in G in the general case. For some groups the anticenter has been determined [1]. Finite groups with a cyclic Sylow subgroup have a nontrivial anticenter. But a suitable product of dihedral groups has nontrivial anticenter and noncyclic Sylow subgroups. So it seems unlikely that a classification of all finite groups with nontrivial anticenter can be given. We show in this paper that for nonabelian nilpotent groups the question reduces to finite p-groups having a self-centralizing element. The investigation of these groups seems to be of independent interest, and we give here some results for groups of low class.

DEFINITION. $RG = \{x \in G \mid \text{for } g \in G,\ xg = gx \text{ implies the group generated by } x \text{ and } g \text{ is cyclic}\}$.

$R_0 G = \{x \in G \mid \text{for } g \in G,\ xg = gx \text{ implies } g \text{ is a power of } x\}$.

The elements of RG are said to have trivial centralizer, the elements of $R_0 G$ are called self-centralizing. The anticenter $AC(G)$ of G is the subgroup generated by RG.

LEMMA 1. *$R_0 G \subseteq RG$. For a subgroup H of G we have $H \cap RG \subseteq RH$. The sets $R_0 G$ and RG are characteristic sets.*

Notation. $N_G H$ is the normalizer of H in G.

$c_G H$ is the centralizer of H in G.

Received April 10, 1967.