ON SOME FORMAL IMBEDDINGS

BY
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In this paper, the reader will find the following theorem: *Let X be a smooth irreducible algebraic scheme over an algebraically closed field k. Assume* $\dim X \geq 2$. *Then an imbedding of X into a projective space P over k is uniquely determined by the formal scheme $\hat{P}$ which is obtained by completing P along X.* (See Th.V, §2, for its precise meaning.) We actually prove that the field of "formal-rational" functions on $\hat{P}$ coincides with the field of rational functions on P. If $k = C$ (the complex number field), this result implies that *for any connected open neighborhood U of X in P in the sense of the usual metric topology, every meromorphic function on U extends to a rational function on the entire P.* (This implication is proven, for instance, by applying the technique of GAGA, due to J. P. Serre, to the infinitesimal neighborhoods of X in P which are complex-analytic spaces.)

A general problem I have in mind may be posed as follows: *Let $\mathbf{Z}$ be a regular irreducible formal scheme over a field k, such that if $\mathbf{I}$ is a defining ideal sheaf of $\mathbf{Z}$ then the subschemes X_ν of $\mathbf{Z}$ defined by $\mathbf{I}^{\nu+1}$ are proper over k. Let $A = H^0(\mathbf{Z}, 0_{\mathbf{Z}})$ which is a k-algebra. We ask if there exists an A-morphism $f : \mathbf{Z} \to T$ with an integral scheme of finite type (or finite presentation) over A such that if $g : \mathbf{Z} \to W$ is any A-morphism into an A-scheme of finite type (or finite presentation), then there exists a unique rational map $h : T \to W$ with $g = hf$.*[2]

In this paper, my interest is confined strictly to the case of ample normal bundle (e.g. $X = X_0$ is smooth and the dual of I/I^2 as a sheaf of 0_X-modules is an ample locally free sheaf on X). We have a satisfactory answer to the above question only in the case of codimension one, i.e., when I/I^2 is an invertible sheaf on X. (See Theorems I, II, III, §1, and Theorems IV*, V*, §2.) The case of higher codimensions is still very little understood. Our result in this case is done only for a very special kind of imbeddings, i.e., imbeddings into a projective space. This seems, however, to throw some encouraging light onto the general problem of higher codimensions. (See Theorems IV, V, §2.) For a certain technical reason we assume $\dim X \geq 2$ throughout this paper. Some novel phenomena as well as generalizations for the case of $\dim X = 1$ will be investigated in a future joint paper with Matsumura.

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[2] The answer is expected to be affirmative in various interesting cases, but not in general. For instance, let $\mathbf{Z}$ be the completion of a line bundle Z over X along the zero section, where X is smooth and projective. The answer is negative if Z is associated with a non-torsion point of the Picard variety $\mathrm{Pic}^0(X)$.