CARTER SUBGROUPS AND FITTING HEIGHTS OF FINITE
SOLVABLE GROUPS

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Let $G$ be a finite solvable group having Fitting height $h$ (as defined in [7] or in §1 below). Let $H$ be a Carter subgroup of $G$ and $l$ be the length of a composition series of $H$. We shall establish the correctness of a conjecture of John Thompson (at the end of [7]) by proving that

\[(0.1) \quad h \leq 10(2^l - 1) - 4l.\]

This is the result of Theorem 8.5 below, and the rest of this paper is a proof of that theorem.

The upper bound for $h$ given by (0.1) is almost certainly too large. The work of Shamash and Shult [6] leads one to conjecture that there is some constant $K$ such that

\[(0.2) \quad h \leq Kl,\]

for all finite solvable groups $G$. The methods of this paper unfortunately cannot give an upper bound whose order of magnitude is less than $2^l$. This is caused by our very naive approach. Essentially we choose a normal subgroup $P$ of prime order in $H$ and a suitable chain $A_1, \ldots, A_h$ of $H$-invariant sections of $G$. Obviously either $P$ centralizes $A_1, \ldots, A_{[h/2]}$ or there exists a subchain $A_k, A_{k+1}, \ldots, A_{k+[h/2]}$ such that $P$ does not centralize $A_k$. In the latter case we construct (and this is the hard part of the proof) an $H$-invariant chain $D_{k+j}, D_{k+j+1}, \ldots, D_{k+[h/2]}$ of sections of $A_k, A_{k+j+1}, \ldots, A_{k+[h/2]}$ (respectively) such that $j$ is bounded and $P$ centralizes each $D_i$. In either case we obtain a chain of length “almost” $h/2$ of sections of $G$ on which $H/P$ acts, and which satisfies suitable axioms so that the process can be repeated (using a normal subgroup of prime order in $H/P$, etc.) Obviously no method based on this process can give an upper bound smaller than $2^l$.

There are many technical complications in the proof due to the difficulty of handling the case $|P| = 3$ (among other things). But basically it is a straightforward application of the methods of Hall and Higman [3]. The few new concepts which are used are grouped together in Sections 1, 2 and 3. They are the notions of Fitting chains (which are the “correct” chains of sections $A_1, \ldots, A_n$ of $G$), of weak equivalence (which is used in place of equivalence in Fitting chains because it is impossible to verify the latter after

Received April 22, 1968.

1While working on this note, the author was a Sloan Research Fellow. He thanks the Alfred P. Sloan Foundation for their support of his research.

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