RETRACTING THREE-MANIFOLDS ONTO FINITE GRAPHS

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1. Introduction and Definitions

A piecewise-linear 3-manifold-with-boundary M is called a *cube-with-handles of genus n* if M is orientable and is a regular neighborhood of a finite connected graph of Euler characteristic $1 - n$. If M is a polyhedral cube-with-handles of genus n in S^3, then $S^3 - \operatorname{Int} M$ is called a *cube-with-holes of genus n*. We call a cube-with-holes N *retractable* if N can be retracted onto a wedge of n simple closed curves, where n is the genus of N. If such a wedge can be chosen in $\operatorname{Bd} N$, then N is *boundary-retractable*.

In [2], Lambert showed that for each $n \geq 2$, there exists N_n, a cube-with-holes of genus n, such that no mapping of N_n onto a cube-with-handles of genus n, H_n, can take $\operatorname{Bd} N_n$ homeomorphically onto $\operatorname{Bd} H_n$. By our Theorem 2, N_n is retractable. It is our purpose here to note that the existence of such a "boundary-preserving" mapping for a cube-with-holes is equivalent to its being boundary-retractable, and to give examples of cubes-with-holes of arbitrary genus $n \geq 3$ which are not even retractable. Our examples also show that the fundamental group G of a cube-with-holes of genus $n \geq 3$ can be residually nilpotent (Theorem 6) and in fact can have $G/G_m \approx F/F_m$ for each $m \geq 1$, where F is free of rank n (see Corollary 5.14.1, page 353 of [3]), and yet G can fail to be a free group. See the definitions below.

We also give (Theorem 5) a necessary condition for a cube-with-holes to be boundary-retractable, and indicate how to apply it to Lambert's example. We are grateful to Joseph Martin for several helpful conversations about this matter. We have not been able to prove the existence or nonexistence of a non-retractable cube-with-holes of genus two, but Theorem 4 gives a criterion in terms of mappings into the torus which may prove useful.

If G is any group, and $a, b \in G$, we denote the *commutator* $a^{-1}b^{-1}ab$ *of a and b* by $[a, b]$. For non-empty subsets A and B of G, $[A, B]$ denotes the subgroup of G generated by the set S of all commutators $[a, b]$, where $a \in A$, $b \in B$. That is, $[A, B]$ is the smallest subgroup of G containing S. We let G_m denote the m^{th} term in the *lower central series of G*. Specifically, $G_1 = G$, $G_2 = [G_1, G]$, and, in general, $G_{m+1} = [G_m, G]$ for each $m \geq 1$. Each G_m is a normal subgroup of G. We call G_2 the *commutator subgroup of G*, and G/G_2 is *G abelianized.* It is also convenient to introduce the notation G_ω for the normal subgroup $\bigcap_{m \geq 1} G_m$. Finally, if $G_{m+1} = 1$, we say that G is *nilpotent of class m*, and if $G_\omega = 1$ we say that G is *residually nilpotent.*

We will be using some known results from the literature. For example, a theorem of Magnus (see pages 311 and 312 of [3]) asserts that any free group

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