ON THE EXISTENCE AND REPRESENTATION OF INTEGRALS

BY

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1. Introduction

Suppose that Ω is a set, R is a non-empty collection of subsets of Ω, and D is the collection of finite non-empty subsets of R to which M belongs only in case M^*, the union of all the members of M, is in R and the members of M are relatively prime in R, i.e., if A and B are in M then there is no non-empty member of R which is contained in both A and B. We will assume that each non-empty A in R contains a point x such that if M is in D and A is in M then no other member of M contains x.

Let $B(\Omega, R)$ denote the closure in the space of functions from Ω to the number-plane which have bounded final sets of the linear space spanned by the characteristic functions of members of R with respect to the supremum norm $|\cdot|$. We will assume that $B(\Omega, R)$ is an algebra. An *integral* on $B(\Omega, R) \times R$ is a function K from $B(\Omega, R) \times R$ to the number-plane such that (1) for each (f, A) in $B(\Omega, R) \times R$, $K[\;\;, A]$ is a linear functional on $B(\Omega, R)$ and $K[f, \;\;]$ is *additive* on R, i.e., $K(f, M^*) = \sum_{H \text{ in } M} K(f, H)$ for each M in D, and (2) there is an additive function λ from R to the non-negative numbers such that $|K(f, A)| \leq |1_A f| \lambda(A)$, for each (f, A) in $B(\Omega, R) \times R$. This paper is concerned with the existence and representation of integrals on $B(\Omega, R) \times R$.

2. Bounded variation

A finite subset M of R is said to *partition* a member A of R provided $M^* = A$. If each of M_1 and M_2 is a finite subset of R then M_2 is said to *refine* M_1 provided that $M_1^* = M_2^*$ and each member of M_2 is contained in some member of M_1. If (A, B) is in $R \times R$ then $[A, B]$ will denote the collection of non-empty members of R which are contained in both A and B. A subset A of Ω is said to be *R-measurable* if for each B in R there is a partition M of B in D such that each H in M is either contained in A or $[H, A] = \emptyset$ and if $[A, B] \neq \emptyset$ then the common part of A and B is the union of those members of M contained in A.

THEOREM 2.1. *If each member of R is R-measurable, each of M_1 and M_2 is in D, and $M_1^* = M_2^*$, then there is a member M of D which refines each of M_1 and M_2 such that each A in M_1 is the union of those members of M contained in A.*

Proof. Let $\{B_p\}_1^n$, be a reversible sequence with final set M_2. There is a sequence $\{N_p\}_0^n$ with values in D such that $N_0 = M_1$ and, for each integer p in $[1, n]$,

(1) N_p is a refinement of N_{p-1} such that each A in N_{p-1} is the union of those members of N_p contained in A, and

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