SOME NON-SOLUBLE FACTORIZABLE GROUPS

BY

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1. Introduction

In this paper we prove the following theorem:

Theorem. *Let G be a finite non-soluble group such that $G = AB$ where A is a cyclic group and B is a metacyclic group. Then $G/S(G) \cong PGL(2, p)$, where p is a prime greater than 3.*

Metacyclic group will mean throughout a finite group all of whose Sylow subgroups are cyclic. $S(G)$ is the maximal soluble normal subgroup of G and $PGL(2, p)$, $PSL(2, p)$ denotes the projective general linear and the projective special linear groups respectively of dimension 2 over a finite field of p elements.

It will be shown in Section 3 that $S(G)$ is not necessarily a direct factor of G.

For any subset T of a group G, $C(T)$, $N(T)$ and $|T|$ denote respectively the centralizer, normalizer and the number of elements in T. The subgroup generated by T will be written $\langle T\rangle$ and a Sylow p-subgroup of G will be called an S_p-subgroup of G. A subgroup H of a group G is called a *T.I.* subgroup if from $x^{-1}Hx \cap H \neq 1$ it follows that $x \in N(H)$. All groups considered will be finite.

2. Proof of the theorem

We note some properties of a metacyclic group G, see for example [9]. G/G' and G' are cyclic groups of co-prime orders and $G' \cap Z(G) = 1$, where $Z(G)$ denotes the center of G.

We begin with two easy lemmas.

Lemma 1. *Let G be a group which satisfies the following conditions:*
(i) *G contains a maximal subgroup B which is metacyclic.*
(ii) *G has no non-trivial normal soluble subgroup.*
(iii) *G has no normal subgroup of index prime to $[G:B]$.*
Then $Z(B) = 1$ and B' is a T.I. subgroup.

Proof. Let $x \in B' \cap B'^{g}$, $g \in G$. If $x \neq 1$, we have $N(\langle x\rangle) \geqq B, B^{g}$ since $\langle x\rangle$ is a characteristic subgroup of B'. Since B is maximal, $N(\langle x\rangle) = B$ by (ii). Hence $B^{g} = B$ and so $g \in B$ by (i) and (ii). Note that only conditions (i) and (ii) are used so far.

Now let $x \in Z(B)$ have prime order p. Then $N(\langle x\rangle) = B$ by (i) and (ii). We have two cases:

(a) An S_p-subgroup of G is not contained in B. Let P be an S_p-subgroup

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