INVARIANT AND EXTENDIBLE GROUP CHARACTERS

BY

I. M. ISAACS

1. Let G be a finite group and let $N \triangle G$. Suppose $\chi \in \text{Irr}(N)$, the set of irreducible complex characters of N and that χ is invariant under the action of G. We seek conditions sufficient to guarantee that χ can be extended to G, in other words that there exists $\theta \in \text{Irr}(G)$ with $\theta \mid N = \chi$. A related question which is considered in §2 is the following. Suppose $N \subseteq H \triangle G$ and that χ can be extended to H. What conditions will guarantee that some extension of χ to H is invariant in G. In this paper we will provide sufficient conditions for both problems provided that N has a normal solvable subgroup N_0 such that $\chi \mid N_0$ is irreducible. In particular, our results apply if N is solvable. Although both of the theorems proved here may be true without this assumption, these proofs depend strongly on solvability.

We begin with some general remarks which are probably well known. (For instance see [1].) Let $\mathfrak{X}$ be a representation of N which affords χ. Since χ is invariant in G, for each $g \in G$ there exists a matrix Y_g such that for all $h \in N$,

$$(*) \qquad Y_g^{-1}\mathfrak{X}(h)Y_g = \mathfrak{X}(g^{-1}hg).$$

Since χ is irreducible, it follows from Schur's lemma that

$$(**) \qquad Y_{g_1} Y_{g_2} = f(g_1, g_2)Y_{g_1 g_2}$$

where f is a function from $G \times G$ into the complex numbers. We may choose the Y_g in the following manner. Pick a transversal T for the cosets of N in G with $1 \in T$ and define Y_t arbitrarily such that $(*)$ is satisfied for $1 \neq t \in T$. Set $Y_1 = I$, the unit matrix. For arbitrary $g \in G$, write $g = tn$ for $t \in T$ and $n \in N$ and put $Y_g = Y_t \mathfrak{X}(n)$.

The function f associated with this choice of the Y_g satisfies

$$f(t_1 n_1, t_2 n_2) = f(t_1, t_2)$$

for all $t_i \in T$ and $n_i \in N$ and because of this f defines a function $\bar{f}$ on $G/N \times G/N$ and $\bar{f}$ is a factor set of G/N. If $\bar{f}$ is a coboundary, then for some function $\bar{\alpha}$ on G/N we have

$$\bar{f}(x, y) = \bar{\alpha}(x)\bar{\alpha}(y)/\bar{\alpha}(xy)$$

for $x, y \in G/N$. This defines a function α on G which is constant on the cosets of N such that

$$f(g_1, g_2) = \alpha(g_1)\alpha(g_2)/\alpha(g_1 g_2).$$

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