A SET OF GENERALIZED NUMBERS SHOWING BEURLING'S THEOREM TO BE SHARP

BY

HAROLD G. DIAMOND[1]

Beurling [1] proved that the prime number theorem holds for generalized (henceforth g-) numbers if $N(x)$, the number of g-integers not exceeding x, satisfies $N(x) = cx + O(x \log^{-\gamma} x)$ for c a positive number and γ a number greater than $\frac{3}{2}$. Further, he showed that this result is sharp by giving an example of a "prime measure" and associated "integer measure" for which $\gamma = \frac{3}{2}$ but for which the prime number theorem is false. However, the measures of Beurling's example are continuous and thus differ from the usual (atomic) counting measures of prime number theory.

We shall give an example of g-primes and g-integers for which the prime number theorem fails but $N(x) = cx + O\{x(\log x)^{-3/2}\}$. Our construction is based on Beurling's example and a method of approximating measures that we have used in [2].

Let $\pi(x)$ be the number of g-primes not exceeding x, and define $\Pi(x) = \sum_{n=1}^{\infty} n^{-1}\pi(x^{1/n})$. Set $li(x) = k + \int_2^x (\log t)^{-1}\, dt$ (k a constant) and define $\tau(x)$ by

$$\tau(x) = \int_1^x \{1 - \cos(\log t)\}(\log t)^{-1}\, dt \quad \text{for } x \geq 1$$

and $\tau(x) = 0$ for $x < 1$.

PROPOSITION. *Let p_r, the r^{th} g-prime, be defined by $p_r = \tau^{-1}(r)$. Then*

$$N(x) = cx + O\{x(\log x)^{-3/2}\},$$

where $c = \exp\{\int_1^\infty t^{-1}(d\Pi - d\tau)(t)\}$, and $\pi(x)/li(x)$ does not have a limit as $x \to \infty$.

Sketch of proof. The number c is finite and positive because $\Pi(t) - \tau(t) = O(t^{1/2})$. Since $\pi(x) = [\tau(x)]$ and

$$\tau(x) = li(x) - \frac{x}{2 \log x}\{\sin(\log x) + \cos(\log x)\} + O(x \log^{-2} x),$$

$\pi(x)/li(x)$ has no limit as $x \to \infty$. We now estimate $N(x)$, noting first that $dN = \exp d\Pi$ [1; p. 257], [2; §3.1]. The exponential is defined by its power series about the origin; the powers of a measure dA are defined by $dA^0 = \delta =$ point mass 1 at 1 and by $(dA)^n = (dA)^{n-1} * dA$ for $n \geq 1$, where $*$ denotes

Received February 14, 1967.

[1] Research supported by an Air Force grant at the Institute for Advanced Study.