QUASI-NORMAL RINGS

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The aim of the present paper is to exploit ideal-theoretic properties of a class of rings, called quasi-normal for lack of a proper designation, which includes integrally closed domains (i.e. normal domains) and Gorenstein rings. Broadly speaking, they are commutative noetherian rings in which principal ideals have a unique representation as an intersection of irreducible ideals. One could also say that quasi-normality occurs when one substitutes "discrete valuation ring" by "one-dimensional Gorenstein ring" in the usual characterization of normal domains. Such rings were first introduced in [4], where a non-intrinsic definition was used and solely for the purpose of studying reflexive modules.

Here we propose, by analogy with the normal case, to describe two classes of ideals which play, in general, interesting roles: the class of reflexive ideals and that of closed ideals. The main tools are Rees' theory of the grade of an ideal [3] and portions of Bass' survey of the basic properties of Gorenstein rings.

1. Quasi-normal rings

Throughout, all rings are commutative and noetherian. Also, unspecified modules are assumed finitely generated. Before we begin our journey we recall some basic definitions. For an irreducible ideal I in a ring it is understood that I is not an intersection of properly larger ideals. In the noetherian case any ideal I can be written as an intersection $J_1 \cap \cdots \cap J_n$ of irreducible ideals without superfluous ones; there might be several such representations but the integer n is invariant. After [3], we say that the ideal I has grade n if it contains an R-sequence of length n but no R-sequence of length $n + 1$. Finally, for basic facts and terminology on commutative noetherian rings, we refer, without mention, to [2]. Leading to our main object are the following

LEMMA 1.1. *Let I and J be primary ideals, P and Q their corresponding primes. If $P \subset Q$ but are distinct, there exists a primary ideal J', properly contained in J, with $I \cap J = I \cap J'$.*

Proof. We can assume $I \cap J = (0)$. Let n be an integer such that $Q^n \subset J$; then $Q^{(n)}$, the n-th symbolic power of Q, is contained in J. Now $Q^{(n+1)} \neq Q^{(n)}$ for otherwise, localizing at Q, $Q_Q^{n+1} = Q_Q^n$ and $Q_Q^n = (0)$ by the Nakayama's lemma. Since height $Q \geq 1$ this is impossible. Now take $J' = Q^{(n+1)}$.

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